MECH 5507
Advanced Kinematics

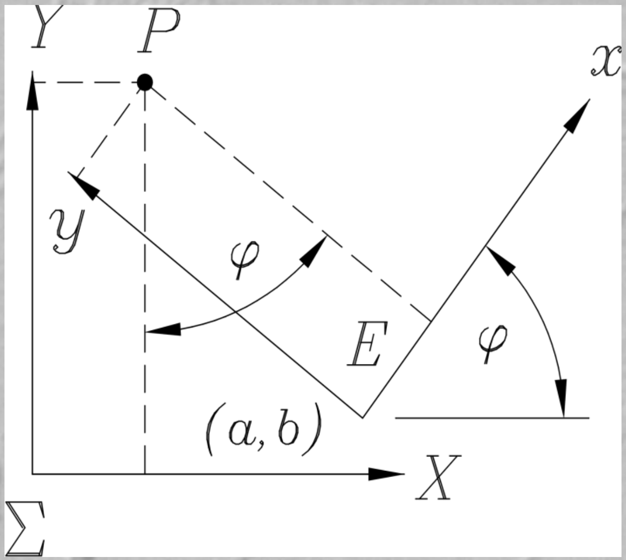
KINEMATIC MAPPING APPLICATIONS

Professor M.J.D. Hayes
Department of Mechanical and Aerospace Engineering

Planar Kinematic Mapping

- Three parameters, a, b and ϕ describe a planar displacement of E with respect to Σ.
- The coordinates of a point in E can be mapped to those of Σ in terms of a, b and ϕ:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & a \\ \sin\varphi & \cos\varphi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



  - (x:y:z): homogeneous coordinates of a point in E.
  - (X:Y:Z): homogeneous coordinates of the same point in Σ.
  - (a,b): Cartesian coordinates of O_E in Σ.
  - ϕ: rotation angle from X- to x-axis, positive sense CCW.

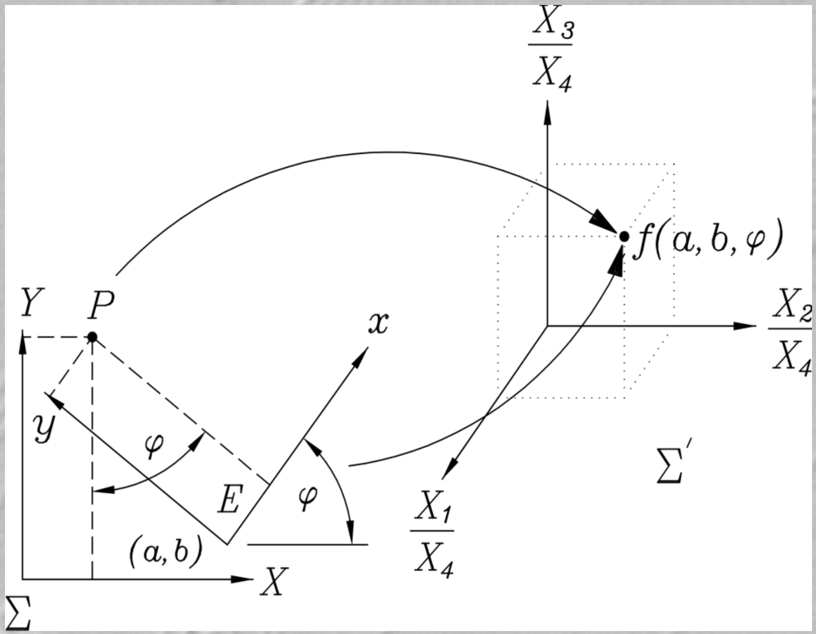
Kinematic Mapping

- The mapping takes distinct poles to distinct points in a 3-D projective image space. It is defined by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} a\sin(\varphi/2) - b\cos(\varphi/2) \\ a\cos(\varphi/2) + b\sin(\varphi/2) \\ 2\sin(\varphi/2) \\ 2\cos(\varphi/2) \end{bmatrix}$$

- Dividing by X_4 normalizes the coordinates:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(a\tan(\varphi/2) - b\right) \\ \frac{1}{2}\left(a + b\tan(\varphi/2)\right) \\ \tan(\varphi/2) \\ 1 \end{bmatrix}$$



- The inverse mapping is:

$$\begin{aligned} \tan(\phi/2) &= X_3/X_4 \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2) \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2) \end{aligned}$$

Kinematic Mapping

- Using half-angle substitutions and these above relations the basic Euclidean group of planar displacements can be written in terms of the image points

$$\lambda\begin{bmatrix}X\\Y\\Z\end{bmatrix}=\begin{bmatrix}X_4^2-X_3^2 & -2X_3X_4 & 2(X_1X_3+X_2X_4)\\ 2X_3X_4 & X_4^2-X_3^2 & 2(X_2X_3-X_1X_4)\\ 0 & 0 & X_3^2+X_4^2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}.$$

- The inverse transformation yields

$$\mu\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}X_4^2-X_3^2 & 2X_3X_4 & 2(X_1X_3-X_2X_4)\\ -2X_3X_4 & X_4^2-X_3^2 & 2(X_2X_3+X_1X_4)\\ 0 & 0 & X_3^2+X_4^2\end{bmatrix}\begin{bmatrix}X\\Y\\Z\end{bmatrix}.$$

- λ and μ being non-zero scaling factors arising from the use of homogeneous coordinates.

Constraint Manifold Equation

- Consider the motion of a fixed point in *E* constrained to move on a fixed circle in Σ, with radius *r*, centred on the homegeneous coordinates $(X_C : Y_C : Z)$ and having the equation

$$K_0(X^2+Y^2)+2K_1XZ+2K_2YZ+K_3Z^2=0,$$

where

$$K_0 = \text{arbitrary homogenising constant.}$$

  - If $K_0 = 1$, the equation represents a circle, and

$$K_1 = -X_C,$$
$$K_2 = -Y_C,$$
$$K_3 = K_1^2 + K_2^2 - r^2.$$

  - If $K_0 = 0$, the equation represents a line with line coordinates

$$[K_1 : K_2 : K_3] = \left[\frac{1}{2}L_1 : \frac{1}{2}L_2 : L_3\right].$$

PR-Dyad Line Coordinates

- For *PR*-dyads the K_i line coordinates are generated by expanding the determinant created from the coordinates of a known point on the line, and the known direction of the line, both fixed relative to Σ:

$$\begin{vmatrix} X & Y & Z \\ F_{X/\Sigma} & F_{Y/\Sigma} & 1 \\ \cos\xi_\Sigma & \sin\xi_\Sigma & 0 \end{vmatrix}$$

where

$$\begin{aligned} X, Y, Z &= \text{homogenious coordinates of points on the line,} \\ F_{X/\Sigma}, F_{Y/\Sigma} &= \text{coordinates of fixed point on the line in } \Sigma, \\ \xi_\Sigma &= \text{angle of the line relative to } \Sigma. \end{aligned}$$

giving

$$[K_1 : K_2 : K_3] = \left[-\frac{1}{2}\sin\xi_\Sigma : \frac{1}{2}\cos\xi_\Sigma : F_{X/\Sigma}\sin\xi_\Sigma - F_{Y/\Sigma}\cos\xi_\Sigma \right].$$

RP-Dyad Line Coordinates

- For *RP*-dyads the K_i line coordinates are generated by expanding the determinant created from the coordinates of a known point on the line, and the known direction of the line, both fixed relative to *E*:

$$\begin{vmatrix} x & y & z \\ M_{x/E} & M_{y/E} & 1 \\ \cos\xi_E & \sin\xi_E & 0 \end{vmatrix}$$

where

$$\begin{aligned} x, y, z &= \text{homogenious coordinates of points on the line}, \\ M_{x/E}, M_{y/E} &= \text{coordinates of a fixed point on the line in } E, \\ \xi_E &= \text{angle of the line relative to } E. \end{aligned}$$

giving

$$[K_1 : K_2 : K_3] = \left[-\frac{1}{2}\sin\xi_E : \frac{1}{2}\cos\xi_E : M_{x/E}\sin\xi_E - M_{y/E}\cos\xi_E\right].$$

Constraint Manifold Equation

- The constraint manifold for a given dyad represents all relative displacements of the dyad links when disconnected from the other two links in a four-bar mechanism.
- An expression for the image space manifold that corresponds to the kinematic constraints emerges when $(X : Y : Z)$, or $(x : y : z)$ from

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & 2X_3X_4 & 2(X_1X_3 - X_2X_4) \\ -2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 + X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

are substituted into

$$K_0(X^2 + Y^2) + 2K_1XZ + 2K_2YZ + K_3Z^2 = 0.$$

Constraint Manifold Equation

- The result is the general image space constraint manifold equation:

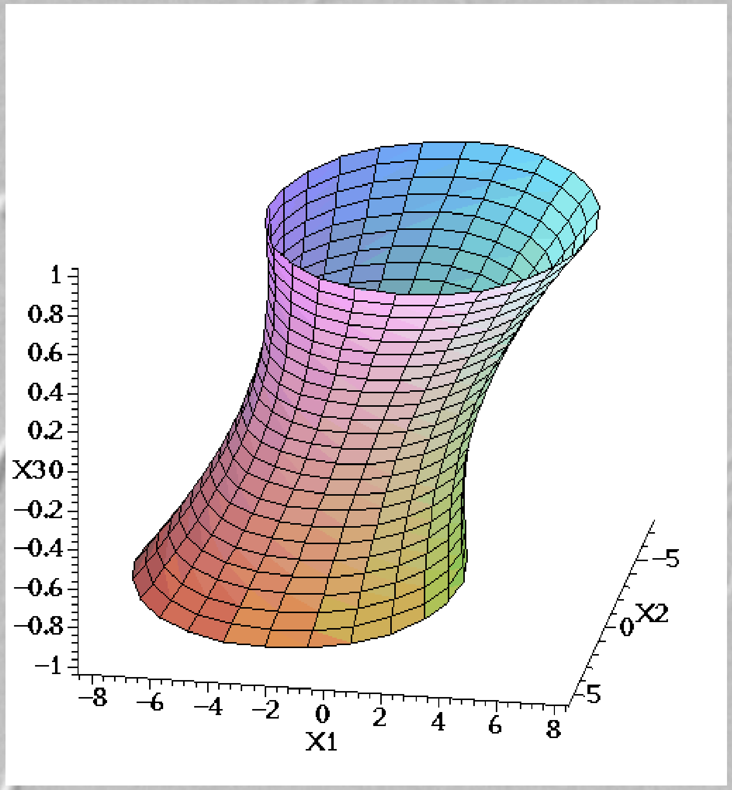
$$CS: K_0(X_1^2 + X_2^2) + \frac{1}{4}(K_0[x^2 + y^2] + K_3 - 2[K_1 x + K_2 y])X_3^2 - (K_1 + K_0 x)X_1 X_3 + (K_2 - K_0 y)X_2 X_3 \mu$$

$$(K_0 y + K_2)X_1 \pm (K_0 x + K_1)X_2 \mu (K_1 y - K_2 x)X_3 +$$

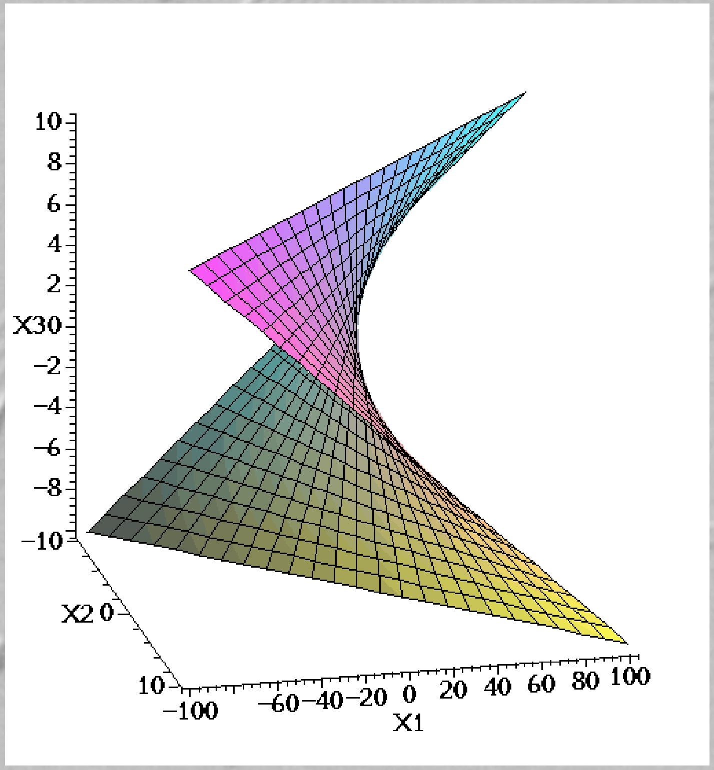
$$\frac{1}{4}(K_0[x^2 + y^2] + K_3 + 2[K_1 x + K_2 y]) = 0.$$

- If the kinematic constraint is
  - a fixed point in E bound to a circle (K_0=1), or line (K_0=0) in Σ, then ($x : y : z$) are the coordinates of the coupler reference point in E and the upper signs apply.
  - a fixed point in Σ bound to a circle (K_0=1), or line (K_0=0) in E, then ($X : Y : Z$) are substituted for ($x : y : z$), and the lower signs apply.

Constraint Manifold Equation



K_0 =1: the *CS* is a skew hyperboloid of one sheet (*RR* dyads).



K_0 = 0: *CS* is an hyperbolic paraboloid (*RP* and *PR* dyads).

SOLVING THE BURMESTER PROBLEM USING KINEMATIC MAPPING

© ***M. John D. Hayes[1], Paul J. Zsombor-Murray[2]***

[1]Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, Canada,
[2]Centre for Intelligent Machines, McGill University, Montréal, Canada

ASME DETC 2002

Mechanisms Synthesis and Analysis Symposium - Special Session:

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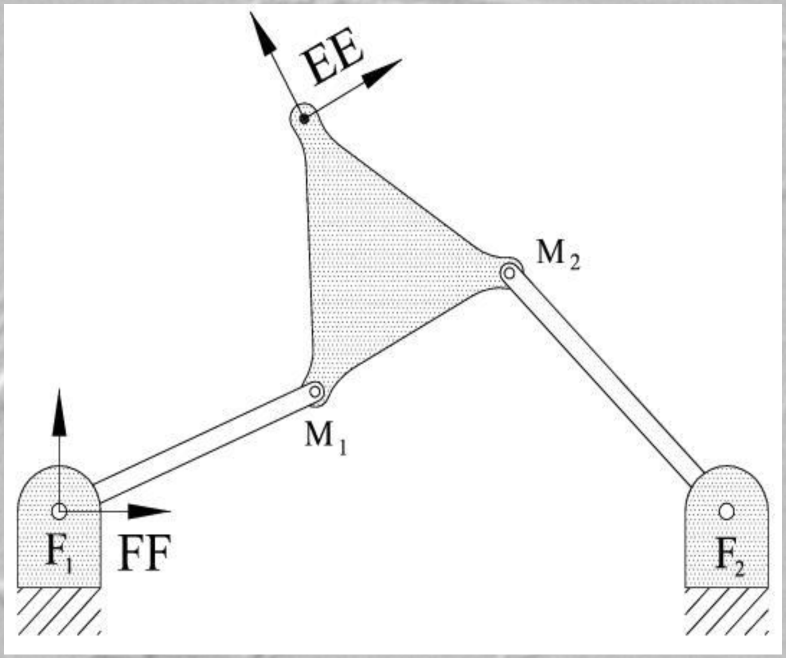
Tuesday October 1, 2002

Five Position Exact Synthesis

- The *five-position Burmester problem* may be stated as:
  - *given five positions of a point on a moving rigid body and the corresponding five orientations of some line on that body, design a four-bar mechanism whose coupler crank pins are located on the moving body and is assemblable upon these five poses.*



- In this example we assume the dyad types we wish to synthesize by setting $K_0=1$, thereby specifying *RR*-dyads.

Nature of the Constraint Surfaces

- Burmester theory states that five poses are sufficient for exact synthesis of two, or four dyads capable of, when pared, producing a motion that takes a rigid body through exactly the five specified poses.
- This means that five non coplanar points in the image space are enough to determine two, or four dyad constraint surfaces that intersect in a curve containing the five image points.
- This is interesting, because, in general, nine points are required to specify a quadric surface (any function $f(x,y,z)=0$ is a surface):

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

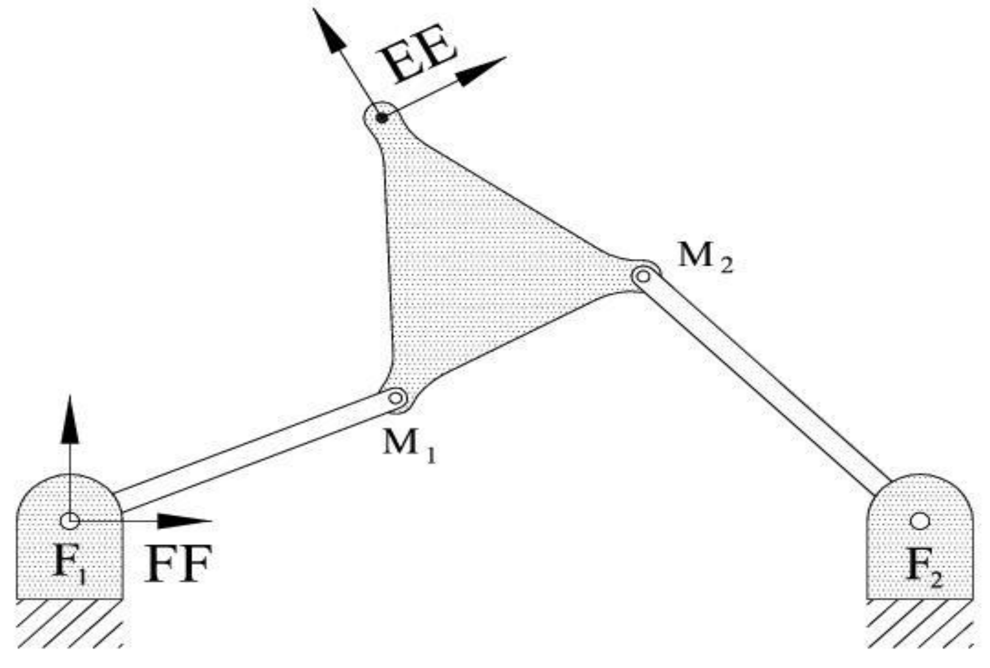
- The equation contains ten *coefficients*; their ratios give nine independent constraints whose values determine the equation.
- It turns out that the special nature of the hyperboloid and hyperbolic paraboloid constraint surfaces represent four constraints on the quadric coefficients; thus five points are sufficient.

Nature of the Constraint Surfaces

- The *RR*-dyad constraint hyperboloids intersect planes parallel to $X_3 = 0$ in circles.
- Thus all constraint hyperboloids contain the image of the imaginary circular points, J_1 and J_2: $(1: \pm i : 0 : 0)$.
- The points J_1 and J_2 are on the line of intersection $X_3 = 0$ and $X_4 = 0$.
- This real line, l, is the axis of a pencil of planes that contain the complex conjugate planes V_1 and V_2, which are defined by $X_3 \pm iX_4 = 0$.
- The *RR*-dyad hyperboloids all have V_1 and V_2 as tangent planes, though not at J_1 and J_2.
- The *PR*- and *RP*-dyad hyperbolic paraboloids contain l as a generator, and therefore also contain J_1 and J_2.
- In addition, V_1 and V_2 are the tangent planes at J_1 and J_2.
- Taken together, these conditions impose four constraints on every constraint surface for *RR*-, *PR*- and *RP*-dyads.
- Thus, only five non coplanar points are required to specify one of these surfaces.

Application to the Burmester Problem

- Goal:
  - determine the moving circle points, M_1 and M_2 of the coupler (revolute centres that move on fixed centred, fixed radii circles as a reference coordinate system, EE, attached to the coupler moves through the given poses).

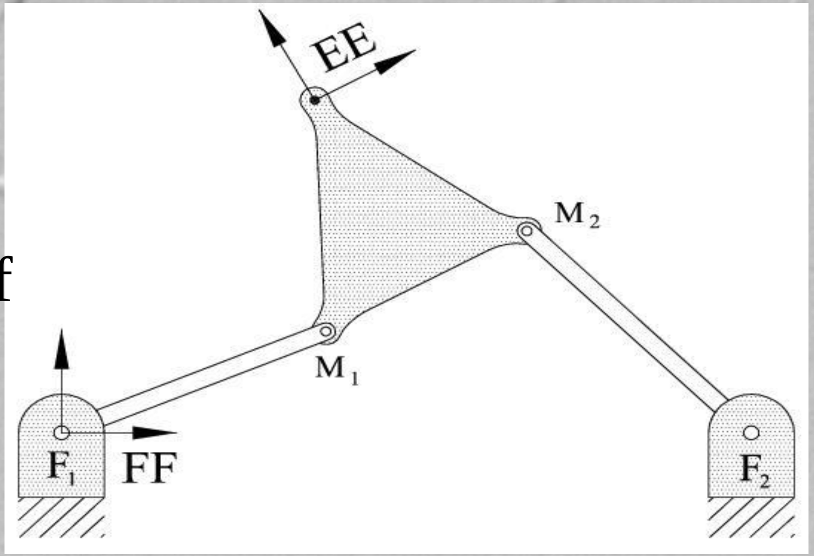


The Five Poses

- To convert specified pose variables a, b, and ϕ to image space coordinates, we first divide through by X_4 to get

$$X_1 = \frac{\left(a\tan(\phi/2) - b\right)}{2}, \quad X_2 = \frac{\left(a + b\tan(\phi/2)\right)}{2}, \quad X_3 = \tan(\phi/2), \quad X_4 = 1.$$

- The five poses are specified as (a_i, b_i, ϕ_i), $i = 1, ..., 5$, the planar coordinates the origin of EE, and orientation all relative to $(0,0,0^o)$ in FF.
- The locations of the origins of FF and EE are arbitrary.



The Five Equations

- We get five simultaneous constraint equations.
- Each represents the constraint surface for a particular dyad.
- This set of equations is expressed in terms of eight variables:
  i. $X_1, X_2, X_3, X_4 = 1$, the dehomogenized coupler pose coordinates in the image space.
  ii. K_1, K_2, K_3, the coefficients of a circle equation ($K_0 = 1$).
  iii. $x, y, z = 1$, coordinates of the moving crank-pin revolute centre, on the coupler, which moves on a circle.
- Since X_1, X_2, X_3, are given, we solve the system for the remaining five variables
  - K_1, K_2, K_3, x, y.

Geometric Interpretation

- The Geometric interpretation is:
  - five given points in space are common to, at most, four RR-dyad hyperboloids of one sheet.
  - If two real solutions result, then all 4*R* mechanism design information is available:

    i. Each circle centre is at $X_C = -K_1,\ Y_C = -K_2$.

    ii. Circle radii are $r^2 = K_3 - (X_C^2 + Y_C^2)$.

    iii. Coupler length is $L^2 = (x_i - x_j)^2 + (y_i - y_j)^2,\ i, j \in \{1,2,3,4\}, i \neq j$.

  - In the case of iii, the subscripts refer to two solutions *i* and *j*.
  - If four real solutions result, the corresponding dyads can be paired in six distinct ways, yielding six 4*R* mechanisms all capable of guiding the coupler through the five specified poses.

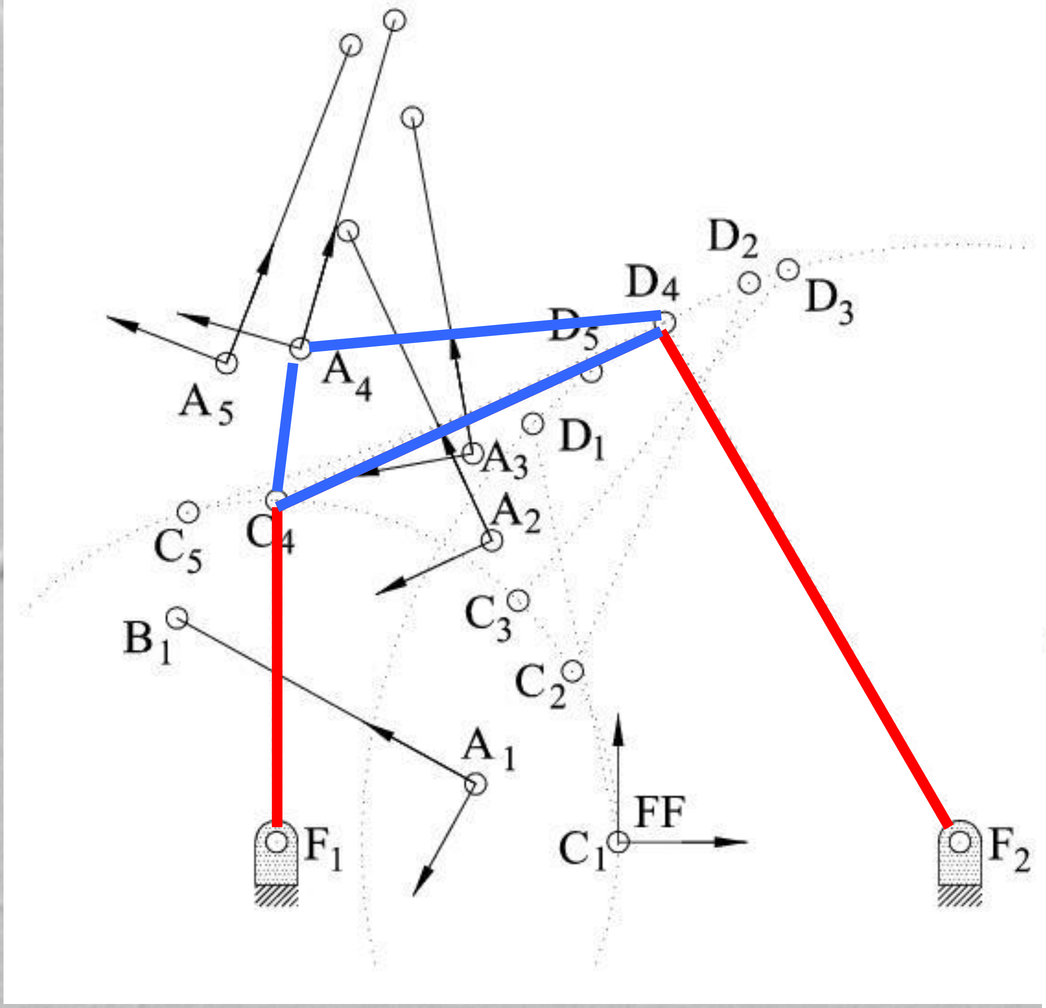
Crank Angles

- To construct the mechanism in its five poses, the crank angles must be determined.
- Take each (x_i, y_i, $z = 1$), and perform the multiplication for each with the five pose variables in

$$\begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix} = \begin{bmatrix} 1-X_3^2 & -2X_3 & 2(X_1X_3+X_2) \\ 2X_3 & 1-X_3^2 & 2(X_2X_3-X_1) \\ 0 & 0 & X_3^2+1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix},$$

- The corresponding sets of (X_i, Y_i) are the Cartesian coordinates of the moving *R*-centres expressed in FF, implicitly define the crank angles.
- For a practical design branch continuity must be checked.

Mechanism to Generate Poses



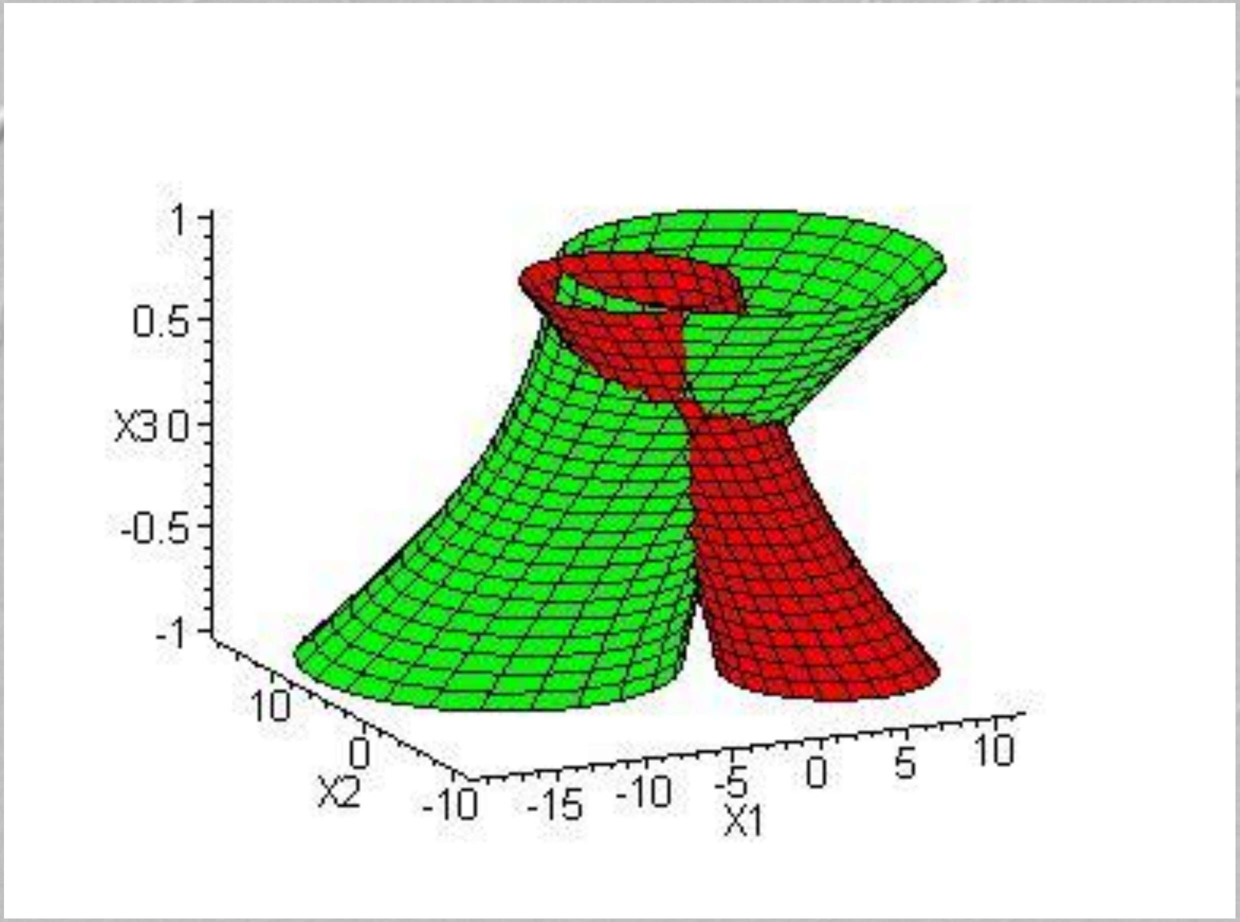
Parameter	Value
F_1	(-8,0)
F_2	(8,0)
F_1F_2	16
F_1C	8
CD	10
DF_2	14

Table 1. THE GENERATING MECHANISM

i^{th} Pose, A_i	a	b	φ (deg)
1	-3.339	1.360	150.94
2	-2.975	7.063	114.94
3	-3.405	9.102	100.22
4	-7.435	11.561	74.07
5	-9.171	11.219	68.65

Table 2. FIVE RIGID BODY POSES IN FF.

The Constraint Hyperboloids



The two constraint hyperboloids for the left and right dyads

Solution

Parameter	Value
F_1	(-7.997,0.001)
F_2	(7.983,-0.023)
F_1F_2	15.980
F_1C	7.999
CD	10.003
DF_2	13.972

Table 4. THE SYNTHESIZED MECHANISM

Parameter	Value
F_1	(-8,0)
F_2	(8,0)
F_1F_2	16
F_1C	8
CD	10
DF_2	14

Table 1. THE GENERATING MECHANISM

Towards Integrated Type and Dimensional Synthesis of Mechanisms for Rigid Body Guidance

© M.J.D. Hayes[1], P.J. Zsombor-Murray[2]

[1]Department of Mechanical and Aerospace Engineering,
Carleton University, Ottawa, Canada

[2]Centre for Intelligent Machines, McGill University,
Montréal, Canada

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Wednesday, June 2, 2004

Integrated Type and Dimensional Synthesis

- Now we try to integrate both *type* and *dimensional* synthesis into one algorithm.
- We shall leave K_0 as an unspecified homogenizing coordinate and solve the five synthesis equations for K_1, K_2, K_3, x, and y in terms of K_0.
- In the solution, the coefficients K_1, K_2, and K_3 will depend on K_0.
- If the constant multiplying K_0 is relatively *very large*, then we will set $K_0 = 0$, and define K_1, K_2, and K_3 as line coordinates proportional to the Grassmann line coordinates:

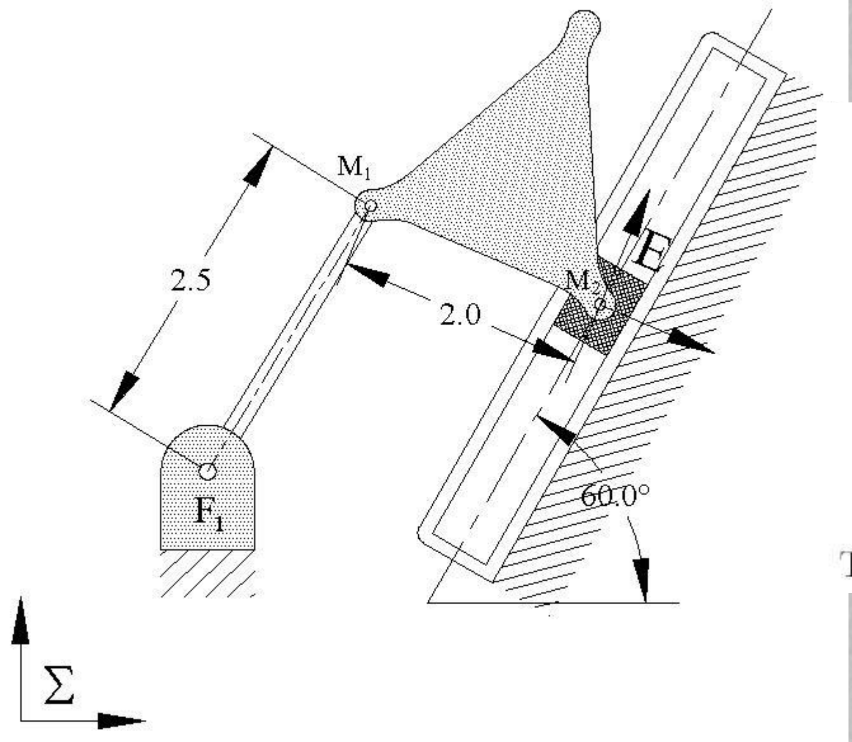
$$[K_1 : K_2 : K_3] = \left[-\frac{1}{2}\sin\xi_\Sigma : \frac{1}{2}\cos\xi_\Sigma : F_{X/\Sigma}\sin\xi_\Sigma - F_{Y/\Sigma}\cos\xi_\Sigma \right].$$

Integrated Type and Dimensional Synthesis

- Otherwise, $K_0 = 1$, and the circle coordinate definitions for K_1, K_2, and K_3 are used:

$$
\begin{aligned}
K_0 &= \text{arbitrary homogenising constant},\\
K_1 &= -X_C,\\
K_2 &= -Y_C,\\
K_3 &= K_1^2 + K_2^2 - r^2.
\end{aligned}
$$

Example



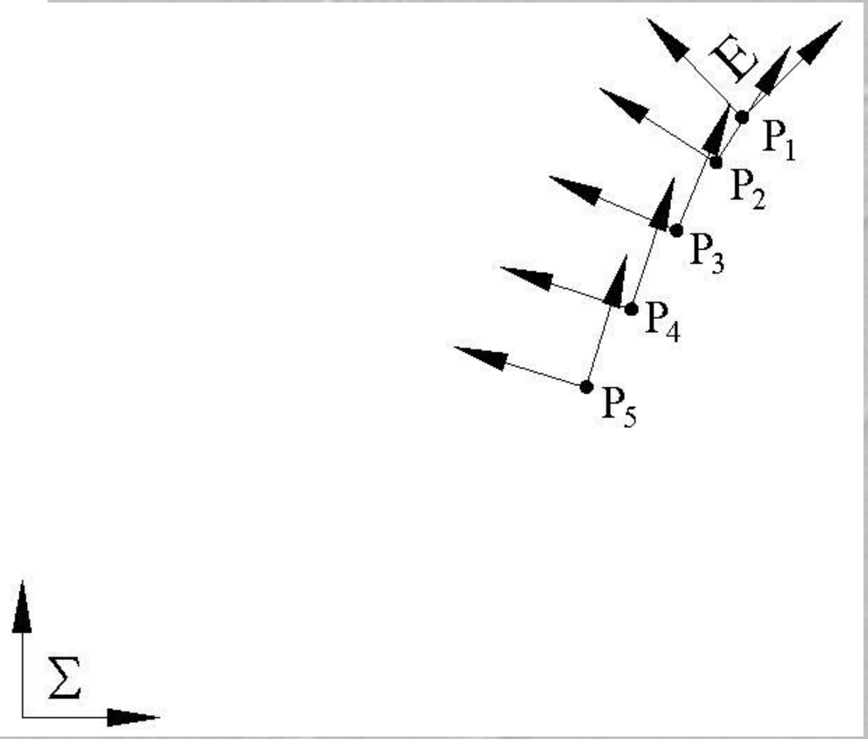
parameter	value
F_1	$(X : Y : Z) = (1.5 : 2 : 1)$
M_1	$(x : y : z) = (-2 : 0 : 1)$
M_2	$(x : y : z) = (0 : 0 : 1)$
M_1M_2	$l = 2$
F_1M_1	$r = 2.5$
P-pair angle	$\vartheta_\Sigma = 60$ (deg)

Table 2: Geometry of the $RRRP$ generating mechanism.

Generated Poses

pose	a	b	φ (deg)
1	5.24080746	4.36781272	43.88348278
2	5.05087057	4.03883237	57.45578356
3	4.76358093	3.54123213	66.99534998
4	4.43453496	2.97130779	72.10014317
5	4.10748142	2.40483444	72.30529428

Table 1: The five desired poses of the $RRRP$ mechanisn



- Convert these pose coordinates to image space coordinates (X_1:X_2:X_3:1), and substitute into the general image space constraint manifold equation.
- This yields five polynomial equations in terms of the K_i, x and y.
- Solving for K_1, K_2, K_3, x and y in terms of the homogenizing circle, or line coordinate K_0 yields:

Solutions

Parameter	Surface 1	Surface 2	Surface 3	Surface 4
K_1	$-1.500K_0$	$-4.2909 \times 10^6 K_0$	$-15.6041K_0$	$-8.3011K_0$
K_2	$-2.0000K_0$	$2.4773 \times 10^6 K_0$	$3.4362K_0$	$-5.0837K_0$
K_3	$-2.5801 \times 10^{-6} K_0$	$2.3334 \times 10^7 K_0$	$107.3652K_0$	$93.4290K_0$
x	-2.0000	8.1749×10^{-7}	0.2281	3.7705
y	3.4329×10^{-7}	-1.3214×10^{-6}	-0.7845	-2.0319

Table 3: The constraint surface coefficients.

- At present, heuristics must be used to select an appropriate *value* for K_0 by comparing the relative magnitudes of K_1 and K_2.
- The coefficients for Surfaces 1, 3, and 4 suggest *RR*-dyads when $K_0=1$.
- The rotation centre for Surface 2 is numerically large : $(4.3 \times 10^6, -2.5 \times 10^6)$.
- The crank radius is about 5×10^6.
- This surface should be recomputed as an hyperbolic paraboloid, revealing the corresponding *PR*-dyad.

PR-Dyad

- The reference point with fixed point coordinates in *E* is the rotation centre of the *R*-pair.
- In a *PR*-dyad, it is clear that this point is constrained to be on the line parallel to the direction of translation of the *P*-pair.
- From the Surface 2 coefficients we have $(x,y)=(8.1749\text{x}10^{-7},-1.3214\text{x}10^{-6})$.
- We could transform these coordinates to Σ using one of the specified poses to obtain the required point coordinates, but they are sufficiently close to 0 to assume they are the origin of moving reference frame *E*.
- The angle of the direction of translation of the *P*-pair relative to the X-axis of Σ is ξ_Σ, and is

$$\xi_\Sigma = \arctan\left(\frac{-K_1}{K_2}\right) = \arctan\left(\frac{4.2909\times 10^6 K_0}{2.4773\times 10^6 K_0}\right) = 60.0^o.$$

Dyads

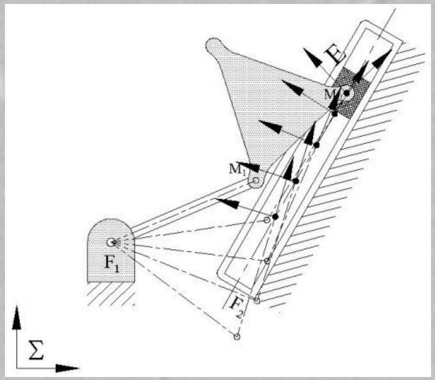
Parameter	Relation	Value
F_1	$(-K_{1_1}, -K_{2_1})$	$(1.500, 2.000)$
M_1	(x_1, y_1)	$(-2.000, 3.4329 \times 10^{-7})$
M_2	(x_2, y_2)	$(8.1749 \times 10^{-7}, -1.3214 \times 10^{-6})$
ϑ_Σ	$\arctan\left(\frac{-K_{1_1}}{K_{2_1}}\right)$	60.0°

Table 4: Geometry of one of six synthesized mechanisms that is identical to the generating $RRRP$ linkage in Figure 1.

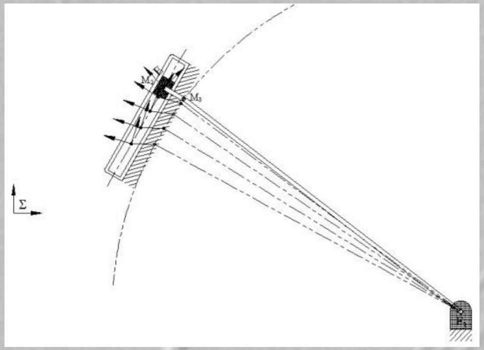
Solution	Dyad surface pairing
1	Dyad 1 - Dyad 2
2	Dyad 2 - Dyad 3
3	Dyad 2 - Dyad 4
4	Dyad 1 - Dyad 3
5	Dyad 1 - Dyad 4
6	Dyad 3 - Dyad 4

Table 5: Dyad pairings yielding the six synthesized mechanisms.

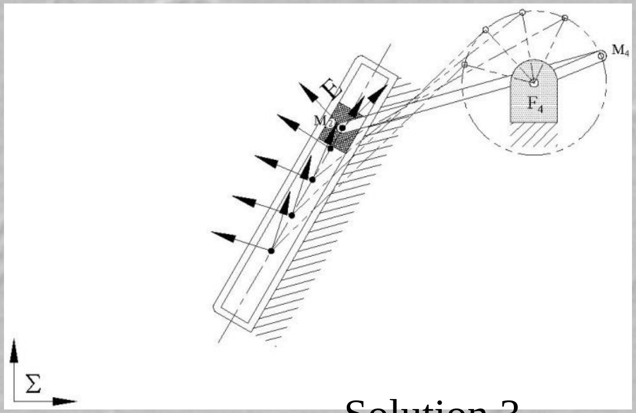
Dyad Pairings



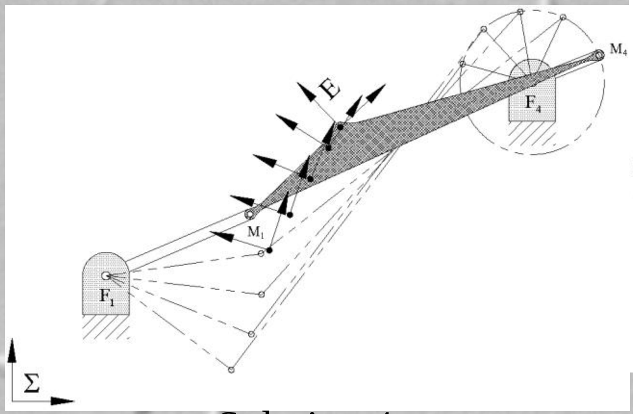
Solution 1



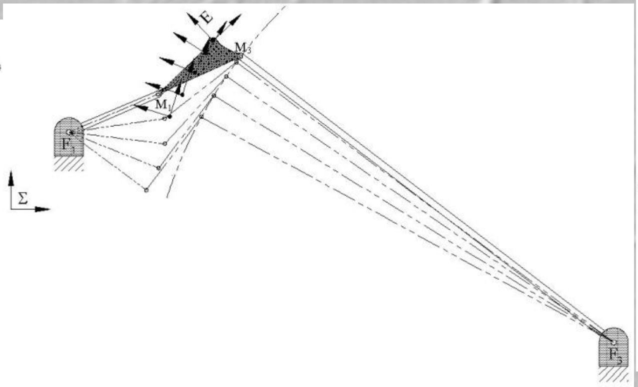
Solution 2



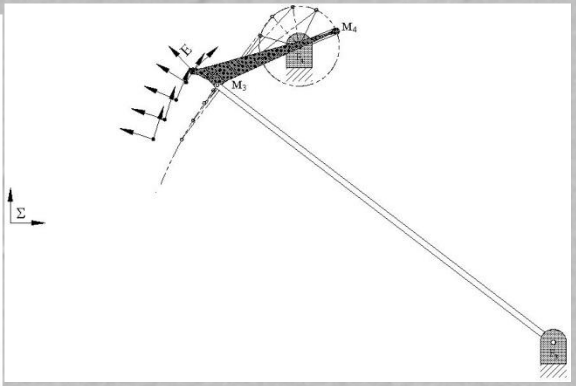
Solution 3



Solution 4



Solution 5



Solution 6

Kinematic Mapping Application to Approximate Type and Dimension Synthesis of Planar Mechanisisms

© M.J.D. Hayes[1], T.Luu[1], X.-W. Chen[2]

[1]Department of Mechanical and Aerospace Engineering,
Carleton University, Ottawa, Canada

[2]School of Computer Science, McGill University,
Montréal, Canada

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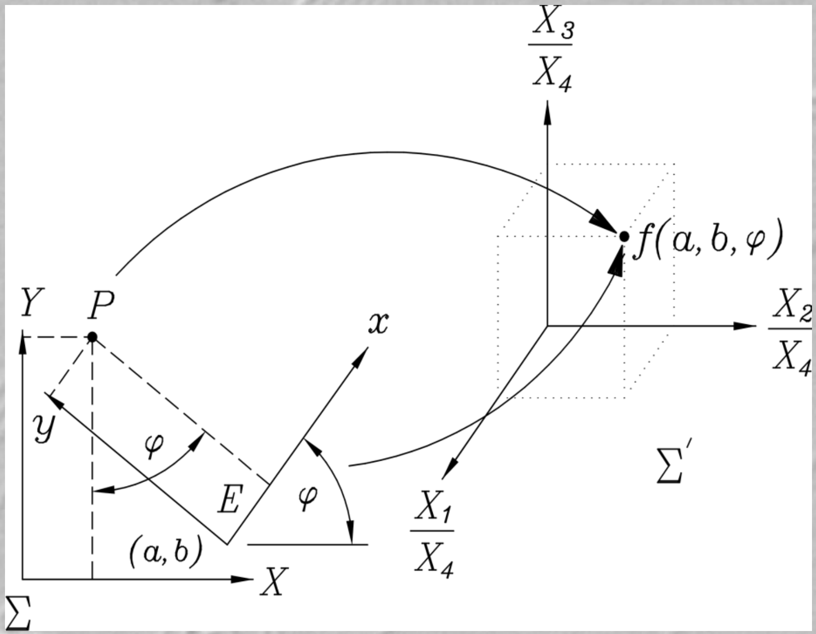
Kinematic Mapping

- The mapping takes distinct poles to distinct points in a 3-D projective image space. It is defined by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} a\sin(\varphi/2) - b\cos(\varphi/2) \\ a\cos(\varphi/2) + b\sin(\varphi/2) \\ 2\sin(\varphi/2) \\ 2\cos(\varphi/2) \end{bmatrix}$$

- Dividing by X_4 normalizes the coordinates:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(a\tan(\varphi/2) - b\right) \\ \frac{1}{2}\left(a + b\tan(\varphi/2)\right) \\ \tan(\varphi/2) \\ 1 \end{bmatrix}$$



- The inverse mapping is:

$$\begin{aligned} \tan(\phi/2) &= X_3/X_4 \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2) \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2) \end{aligned}$$

Kinematic Mapping

- Using half-angle substitutions and these above relations the basic Euclidean group of planar displacements can be written in terms of the image points

$$\lambda\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- λ being non-zero scaling factors arising from the use of homogeneous coordinates.

Circle and Line Coordinates

- Consider the motion of a fixed point in E constrained to move on a fixed circle in Σ, with radius r, centred on the homegeneous coordinates $(X_C : Y_C : Z)$ and having the equation

$$K_0(X^2+Y^2)+2K_1XZ+2K_2YZ+K_3Z^2=0,$$

where

$$K_0 = \text{arbitrary homogenising constant.}$$

  – If $K_0 = 1$, the equation represents a circle, and

$$K_1 = -X_C,$$
$$K_2 = -Y_C,$$
$$K_3 = K_1^2 + K_2^2 - r^2.$$

  – If $K_0 = 0$, the equation represents a line with line coordinates

$$[K_1 : K_2 : K_3] = \left[\frac{1}{2}L_1 : \frac{1}{2}L_2 : L_3\right].$$

Constraint Manifold Equation

- The constraint manifold for a given dyad represents all relative displacements of the dyad.
- An expression for the image space manifold that corresponds to the kinematic constraints emerges when $(X : Y : Z)$ from

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

are substituted into

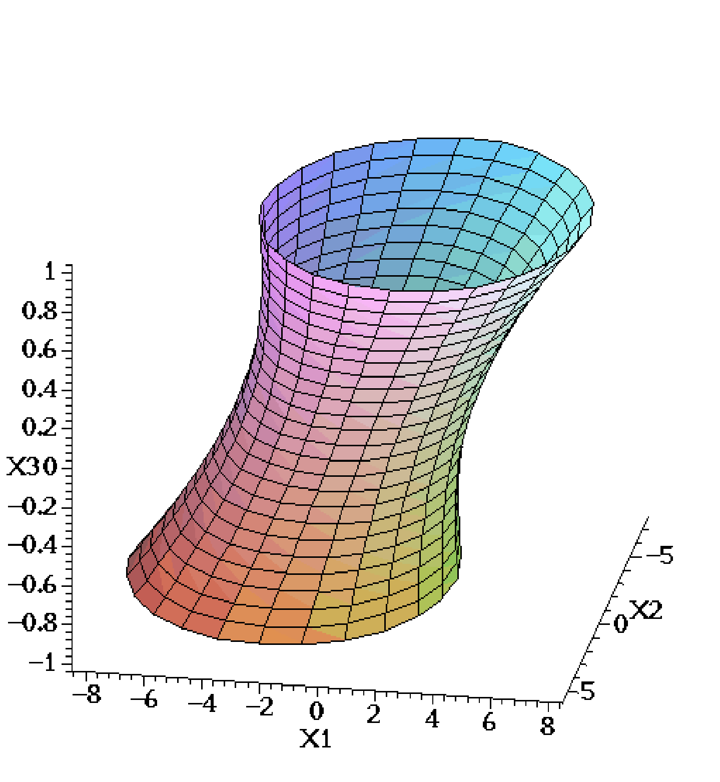
$$K_0(X^2 + Y^2) + 2K_1XZ + 2K_2YZ + K_3Z^2 = 0.$$

Constraint Manifold Equation

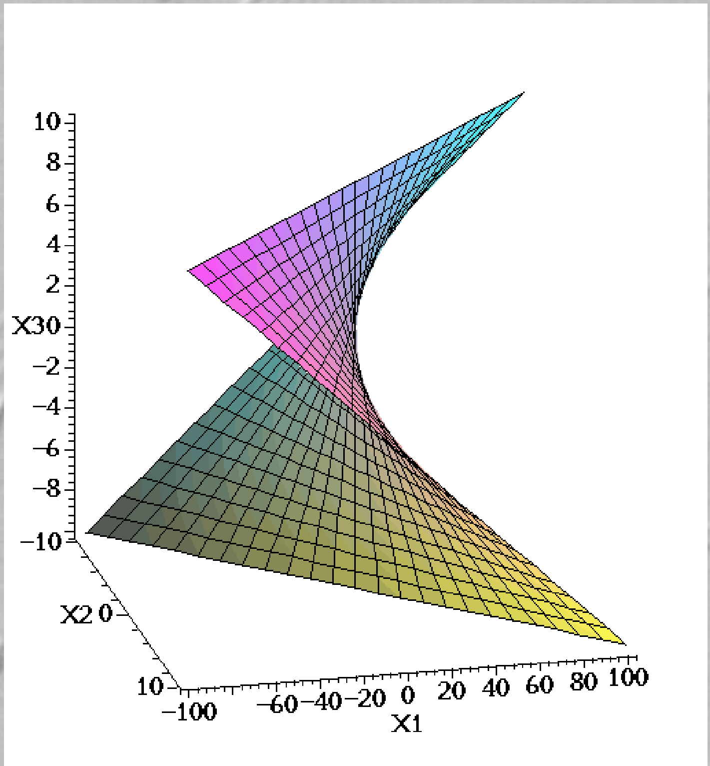
- The result is the general image space constraint manifold equation:

$$CS: K_0(X_1^2 + X_2^2) + \frac{1}{4}(K_0[x^2 + y^2] + K_3 - 2[K_1x + K_2y])X_3^2 - (K_1 + K_0x)X_1X_3 +$$
$$(K_2 - K_0y)X_2X_3 - (K_0y + K_2)X_1 + (K_0x + K_1)X_2 - (K_1y - K_2x)X_3 +$$
$$\frac{1}{4}(K_0[x^2 + y^2] + K_3 + 2[K_1x + K_2y]) = 0.$$

Constraint Manifold Equation



K_0 =1: the *CS* is a skew hyperboloid of one sheet (*RR* dyads).



K_0 = 0: *CS* is an hyperbolic paraboloid (*RP* and *PR* dyads).

SVD

- Any *m* x *n* matrix **C** can be decomposed into the product

$$\mathbf{C}_{m\times n} = \mathbf{U}_{m\times m}\mathbf{S}_{m\times n}\mathbf{V}^{T}_{n\times n}$$

  - where **U** is an orthogonal matrix ($\mathbf{U}\mathbf{U}^T=\mathbf{I}$),
  - the uppermost $n \times n$ elements of **S** are a diagonal matrix whose elements are the singular values of **C**,
  - **V** is an orthogonal matrix ($\mathbf{V}\mathbf{V}^T=\mathbf{I}$).
- The singular values, s_i, of **C** are related to its eigenvalues, λ_i. If **C** is rectangular $\mathbf{C}^\mathrm{T}\mathbf{C}$ is positive semidefinite with non-negative eigenvalues:

$$(\mathbf{C}^T\mathbf{C})\mathbf{x} = \lambda\mathbf{k} \;\Rightarrow\; \left(\mathbf{C}^T\mathbf{C} - \lambda\mathbf{I}\right)\mathbf{k} = 0$$

and $s_i = \sqrt{\lambda_i}$

SVD

- SVD explicitly constructs orthonormal bases for the nullspace and range of a matrix.
  - The columns of **U** whose same-numbered elements s_i are non-zero are an orthonormal set of basis vectors spanning the range of **C**.
  - The columns of **V** whose same-numbered elements s_i are zero are an orthonormal set of basis vectors spanning the nullspace of **C**.
- If $\mathbf{C}_{m \times n}$ does not have full column rank then the last n-rank(**C**) columns of **V** span the nullspace of **C**.
- Any of these columns, in any linear combination, is a non-trivial solution to

$$\mathbf{Ck} = \mathbf{0}$$

Aside: Line and circle feature extraction

- Not specifying a value for K_0 gives a homogeneous linear equation in the K_i:

$$K_0(X^2+Y^2)+2K_1XZ+2K_2YZ+K_3Z^2=0.$$

- It is homogeneous in the projective geometric sense, and homogeneous in the linear algebraic sense in that the constant term is 0:

$$\mathbf{k}\mathbf{X}^T=0.$$

- Four points in the plane yields the following homogeneous system of linear equations:

$$\mathbf{Xk}=\begin{bmatrix} X_1^2+Y_1^2 & 2X_1Z & 2Y_1Z & Z^2 \\ X_2^2+Y_2^2 & 2X_2Z & 2Y_2Z & Z^2 \\ X_3^2+Y_3^2 & 2X_3Z & 2Y_3Z & Z^2 \\ X_4^2+Y_4^2 & 2X_4Z & 2Y_4Z & Z^2 \end{bmatrix}\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}=\mathbf{0}$$

Line and circle feature extraction

- In the general case where **X** has full rank the system has either
  - only the trivial solution, **k**=**0**, or
  - infinitely many nontrivial solutions in addition to the trivial solution.
- Not very useful for feature identification if **k** characterizes the feature.
- However, if the points are all on a line or a circle, then **X** becomes rank deficient by 1.
- In other words, **X** acquires a nullity of 1: the dimension of the nullspace is 1 and is spanned by a single basis vector.
- Since the singular values are lower bounded by 0 and arranged in descending order on the diagonal of **S** by the SVD algorithm a nontrivial solution for **k** is the same numbered column in **V** corresponding to $s_i=0$.
- This is true for any $\mathbf{X}_{m\mathrm{x}4}$, where m≥4, having a nullity of 1.

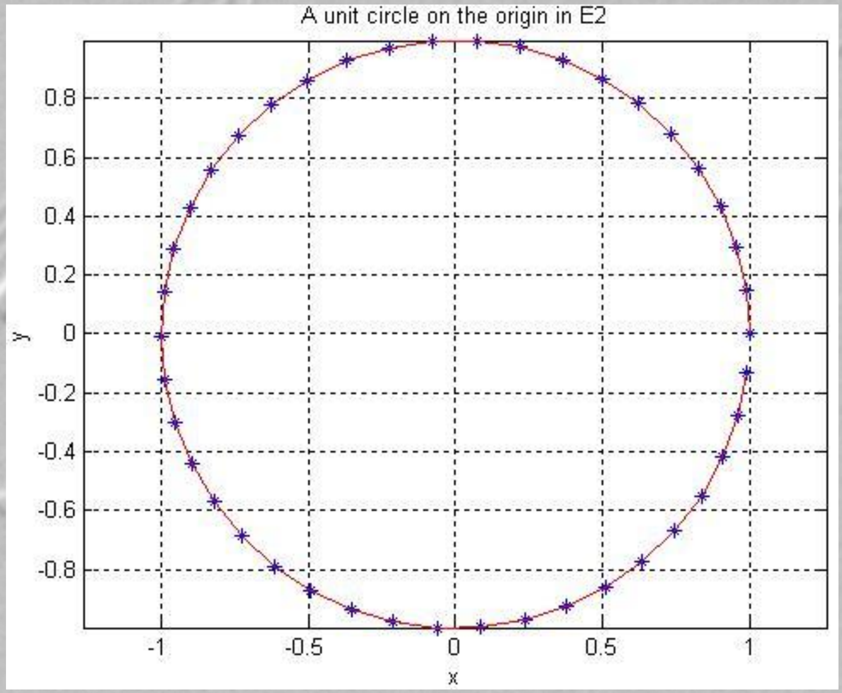
Points Falling Exactly on a Circle

- Given 42 points falling exactly on the unit circle centred on the origin generated by the parametric equations

$$X = r\cos\vartheta$$
$$Y = r\sin\vartheta$$

- This gives rank($\mathbf{X}_{42x4}$)=3
- We have s_4=0 and look at the 4[th] column of **V**:

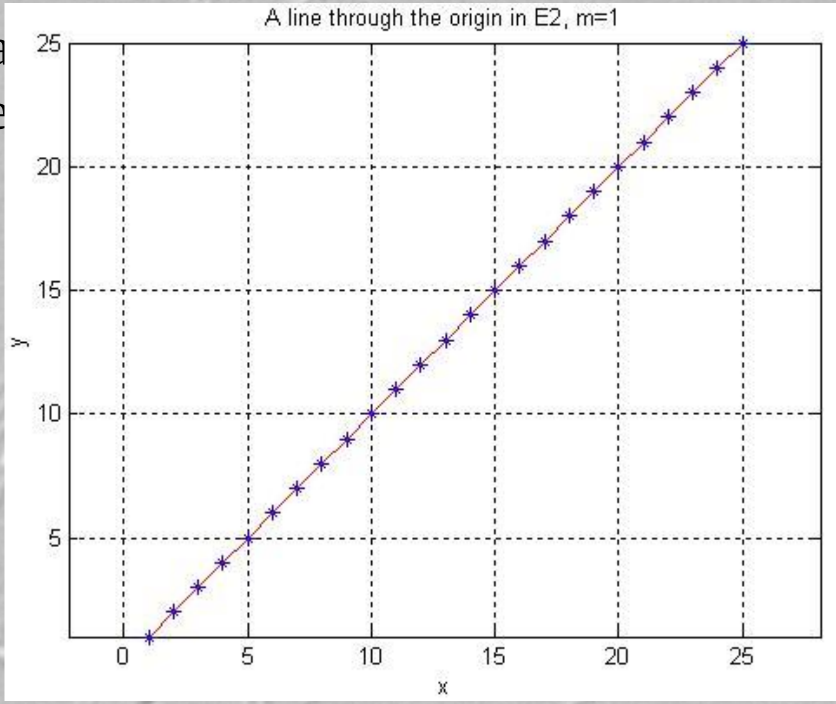
$$\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \text{circle} \\ -X_c \\ -Y_c \\ K_1^2 + K_2^2 - r^2 \end{bmatrix} \Rightarrow \begin{bmatrix} X_c \\ Y_c \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Points Falling Exactly on a Line

- Given 25 points falling exactly on a line through the origin having slope *m*=1, generated by the parametric equations

$$
\begin{aligned}
X &= t \\
Y &= t
\end{aligned}
$$



- This gives rank($\mathbf{X}_{25x4}$)=3
- We have s_4=0 and look at the 4[th] column of **V**:

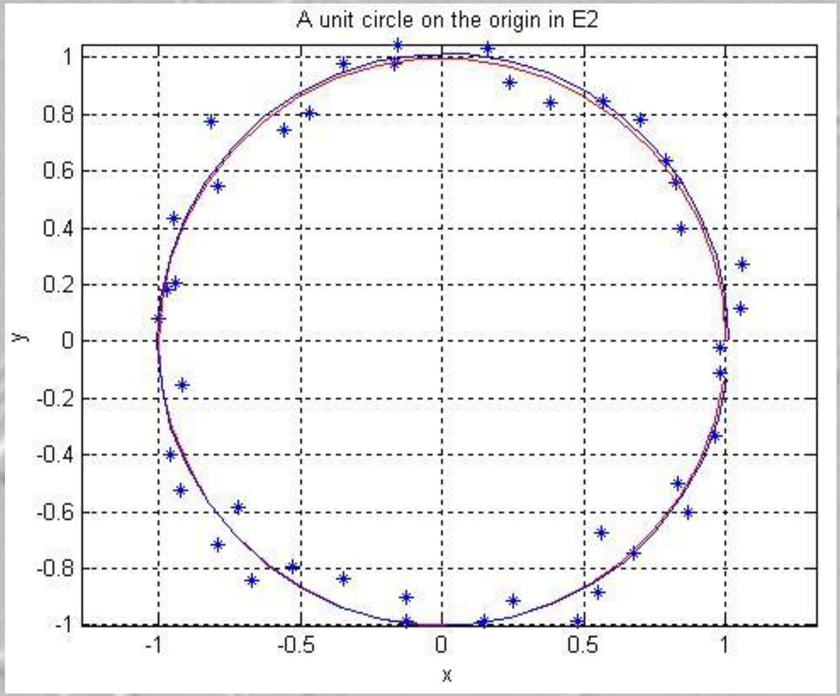
$$
\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7071 \\ -0.7071 \\ 0 \end{bmatrix} = \begin{bmatrix} \text{line} \\ -\frac{1}{2}\sin\xi \\ \frac{1}{2}\cos\xi \\ X\sin\xi - Y\cos\xi \end{bmatrix} \Rightarrow \begin{bmatrix} \xi \\ X \\ Y \end{bmatrix} = \begin{bmatrix} 45^{o} \\ 0 \\ 0 \end{bmatrix}
$$

Points Falling Approximately on a Circle

- Given 42 points falling approximately on the unit circle centred on the origin generated by the parametric equations

$$X = r\cos\vartheta + \text{noise}$$
$$Y = r\sin\vartheta + \text{noise}$$

- This gives rank($\mathbf{X}_{42x4}$)=4 and cond($\mathbf{X}_{42x4}$)=225.2.
- Still,when we look at **V**(:,4):

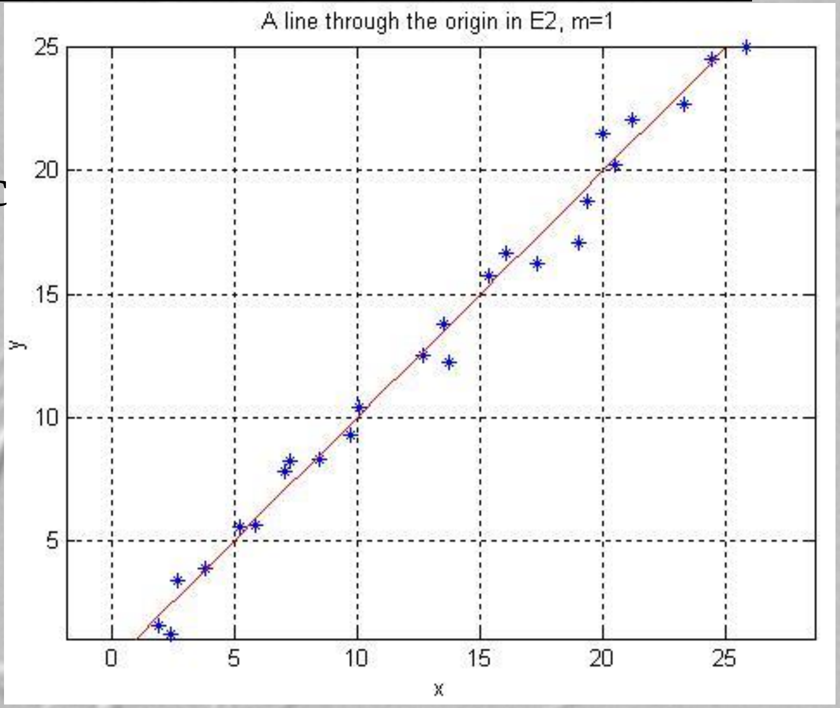


$$\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.00047 \\ 0.00027 \\ -0.99746 \end{bmatrix} = \begin{bmatrix} \text{circle} \\ -X_c \\ -Y_c \\ K_1^2 + K_2^2 - r^2 \end{bmatrix} \Rightarrow \begin{bmatrix} X_c \\ Y_c \\ r \end{bmatrix} = \begin{bmatrix} 0.00047 \\ -0.00027 \\ 0.99746 \end{bmatrix}$$

Points Falling Approximately on a Line

- Given 25 points falling approximately on a line through the origin having slope *m*=1, generated by the parametric equations

$$\begin{aligned} X &= t + \text{noise} \\ Y &= t + \text{noise} \end{aligned}$$



- This gives rank($\mathbf{X}_{25x4}$)=4 and cond($\mathbf{X}_{25x4}$)=5448.6
- Still,when we look at **V**(:,4):

$$\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} -0.00068 \\ 0.71156 \\ -0.69558 \\ -0.00099 \end{bmatrix} = \begin{bmatrix} \text{approximate line} \\ -\frac{1}{2}\sin\xi \\ \frac{1}{2}\cos\xi \\ X\sin\xi - Y\cos\xi \end{bmatrix} \Rightarrow \begin{bmatrix} \xi \\ X \\ Y \end{bmatrix} \cong \begin{bmatrix} 45^{\circ} \\ 0 \\ 0 \end{bmatrix}$$

Approximate Mechanism Synthesis

- To exploit the ability of SVD to construct the basis vectors spanning the nullspace of the homogeneous system of synthesis equations **Ck**=**0**, we must rearrange the terms in the general constraint surface equation, and for now, restrict ourselves to RR- and PR-dyads.
- We obtain a constraint equation linear in the surface shape parameters K_0, K_1, K_2, K_3, and products with x and y:

$$\left[\frac{1}{4}(X_3+1)x^2 + (X_2 - X_1X_3)x + \frac{1}{4}(X_3+1)y^2 - (X_1 + X_2X_3)y + X_2^2 + X_1^2\right]K_0 +$$

$$\left[\frac{1}{2}(1-X_3^2)x - X_3y + X_1X_3 + X_2\right]K_1 + \left[X_3x + \frac{1}{2}(1-X_3^2)y - X_1 + X_2X_3\right]K_2 + \frac{1}{4}(X_3^2+1)K_3 = 0.$$

Approximate Mechanism Synthesis

- There are 12 terms. The X_i are assembled into the m x 12 coefficient matrix **C**.
- The corresponding vector **k** of shape parameters is:

$$\begin{bmatrix} K_0 & K_1 & K_2 & K_3 & K_0x & K_0y & K_0x^2 & K_0y^2 & K_1x & K_1y & K_2x & K_2y \end{bmatrix}^T$$

- Several of the elements of **k** have identical coefficients in **C**:
  - $\frac{1}{4}\left(1+X_3^2\right)$ is the coefficient of K_0x^2, K_0y^2, and K_3.
  - $\frac{1}{2}\left(1-X_3^2\right)$ is the coefficient of K_1x and K_2y.
  - X_3 is the coefficient of K_2x and K_1y.

Approximate Mechanism Synthesis

- The like terms may be combined yielding an m x 8 coefficient matrix $\mathcal{C}$ whose elements are:

$$\left[X_1^2 + X_2^2 \quad X_2 + X_1 X_3 \quad X_2 X_3 - X_1 \quad X_2 - X_1 X_3 \quad -\left(X_1 + X_2 X_3\right) \quad \frac{1}{4}\left(1 + X_3^2\right) \quad \frac{1}{2}\left(1 - X_3^2\right) \quad X_3 \right]$$

- The corresponding 8 x 1 vector $\mathcal{k}$ of shape parameters is:

$$\left[K_0 \quad K_1 \quad K_2 \quad K_0 x \quad K_0 y \quad K_0\left(x^2 + y^2\right) + K_3 \quad \left(K_1 x + K_2 y\right) \quad \left(K_2 x - K_1 y\right) \right]^{\mathrm{T}}$$

- We now have a system of m homogeneous equations in the form

$$\mathcal{C}\mathcal{k} = \mathbf{0}$$

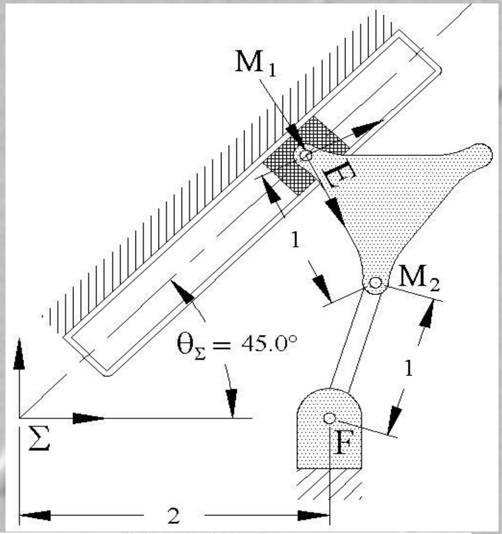
Approximate Mechanism Synthesis

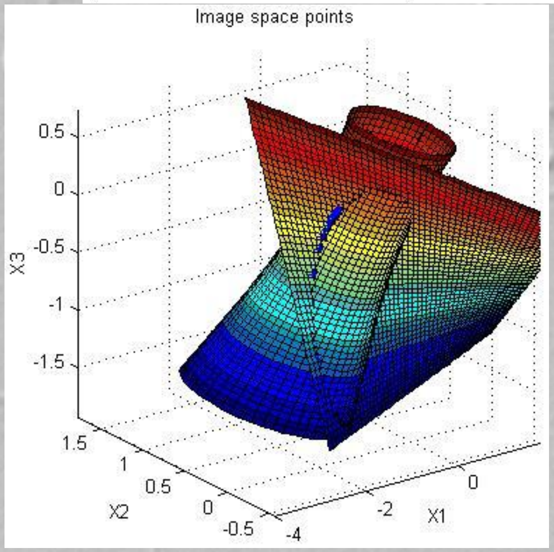
- We obtain the following correspondence between rank($\mathcal{C}$), the mechanical constraints, and the order of the coupler curve:

rank($\mathcal{C}$)	constraint	coupler curve order
8	general planar motion	??
6	two RR-dyads	6
6	one PR-, one RR-dyad	4
5	two PR-dyads	2

- In general, rank($\mathcal{C}$) = 8, with O_E on neither a line or circle.
- Practical application of the approach will require fitting constraint surfaces to their approximate curve of intersection, which means rank($\mathcal{C}$) = 8.
- We will have to approximate $\mathcal{C}$ by matrices of lower rank.
- To start we will investigate Eckart-Young-Mirsky theory.

Example





- An exploratory experiment was devised.
- A PRRR mechanism was used to generate a set of 20 coupler positions and orientations using the origin of E, given by the coordinates (*a*,*b*), as the coupler point, and taking its orientation to be that of the coupler.
- The positions range from (2,1) to (3,2), and the orientations from -5° to -90º.
- The range of motion of the PR- and RR-dyads map to a hyperbolic paraboloid and hyperboloid of one sheet, respectively.
- These quadrics intersect in a spatial quartic, such that rank($\mathcal{C}$) = 6.

Example

- When the rank of a 20 x 8 matrix is deficient by 2, then 2 columns are linear combinations of the remaining 6.
- The column vectors **V**(:,6) and **V**(:,7) in the SVD of $\mathcal{C}$ span its nullspace.
- Any linear combination **V**(:,6) + λ**V**(:,7) is a solution.
- But, we can regard this in a different way.
- We can combine these columns of $\mathcal{C}$ and corresponding elements of $\mathcal{k}$.
- The rank of $\mathcal{C}$ is invariant under this process.
- We obtain two different 20 x 7 coefficient matrices possessing rank = 6.
- The resulting two nullspace vectors represent the generating PR-, and RR-dyads, exactly.

PR-Dyad Synthesis

- To extract the PR-dyad we set K_0=0.
- Recall

$$\mathcal{k} = \begin{bmatrix} K_0 & K_1 & K_2 & K_0 x & K_0 y & K_0\left(x^2+y^2\right)+K_3 & \left(K_1 x + K_2 y\right) & \left(K_2 x - K_1 y\right) \end{bmatrix}^{\mathrm{T}}$$

- In the system $\mathcal{C}\mathcal{k}$ = **0** we can add columns 4 and 5 of $\mathcal{C}$ because K_0=0.
- The resulting 20 x 7 matrix $\mathcal{C}$ possesses rank 6.
- The 7th column of the **V** matrix that results from the SVD of $\mathcal{C}$ yields $\mathcal{k}$ that exactly represents the constraint surface for the generating PR-dyad.

RR-Dyad Synthesis

- To extract the RR-dyad we add columns 2 and 3 of $\mathcal{C}$.
- This can be done when $(X_1 - X_2 X_3)/(X_1 X_3 + X_2)$ has the same scalar value for every, X_1, X_2, and X_3 in the pose data.
- The scalar is the ratio K_1/K_2 of the PR-dyad parameters.
- This happens only when PR-dyad design parameters contain

$$K_3 = x = y = 0$$

- In this case the hyperbolic paraboloid has the equation

$$K_1(X_1 X_3 + X_2) + K_2(X_2 X_3 - X_1) = 0$$

- The curve of intersection with any RR-dyad constraint hyperboloid will be symmetric functions of X_3 in X_1 and X_2.
- An image space curve with rank($\mathcal{C}$) = 6 but PR-dyad design parameters

$$K_3 \neq x \neq y \neq 0$$

can always be transformed to one symmetric in X_1 and X_2.

Results

Table 1. Nullspace vectors obtained by adding two columns of $\mathcal{C}$, and same-numbered elements of $\boldsymbol{\kappa}$

Column 4+5	*Value*	*Column 2+3*	*Value*	*Value/K_0*
K_0	0	K_0	-0.2085	1
K_1	0.7071	$K_1 + K_2$	0.2085	-1
K_2	-0.7071	$K_0 x$	-0.2085	1
$K_0(x+y)$	0	$K_0 y$	0	0
$K_0(x^2+y^2)+K_3$	0	$K_0(x^2+y^2)+K_3$	-0.8340	4
$K1x + K2y$	0	$K1x + K2y$	0.4170	-2
$K2x - K1y$	0	$K2x - K1y$	0	0

Table 2. Generating mechanism shape parameters.

Parameter	*PR-dyad*	*RR-dyad*
K_0	0	1
K_1	-1	-2
K_2	1	0
K_3	0	3
x	0	1
y	0	0

Conclusions and Future Work

- We have presented preliminary results that will be used in the development of an algorithm combining type and dimensional synthesis of planar mechanisms for *n*-pose rigid body guidance.
- This approach stands to offer the designer *all* possible linkages that can attain the desired poses, not just 4R's and not just slider-cranks, but *all* four-bar linkages.
- The results are preliminary, and not without unresolved conceptual issues.
  - Cope with *noise*: random noise greater than 0.01% is problematic.
  - Establish how to proceed with 4R mechanisms.
  - For the general approximate case with rank($\mathcal{C}$) = 8, determine how to approximate $\mathcal{C}$ with lower rank matrices.
  - Establish optimization criteria.
  - Investigate meaningful metrics in the kinematic mapping image space.

Integrated Type And Dimensional Synthesis of Planar Four-Bar Mechanisms

© Tim J. Luu[1] and M.J.D. Hayes[2]

[1]Neptec Design Group Ltd., Ottawa, Canada

[2]Department of Mechanical and Aerospace Engineering,
Carleton University, Ottawa, Canada,

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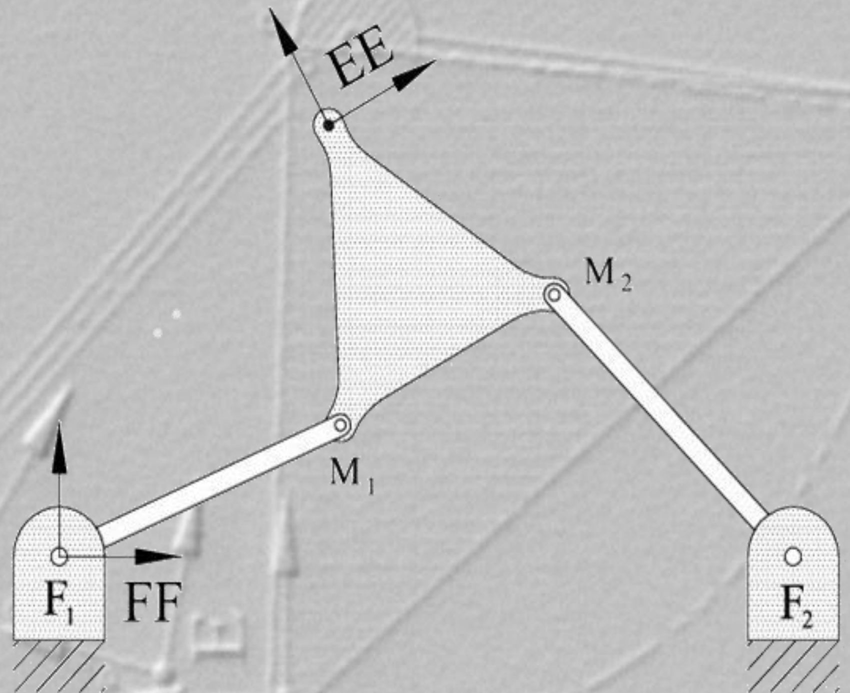
Five Position Exact Synthesis

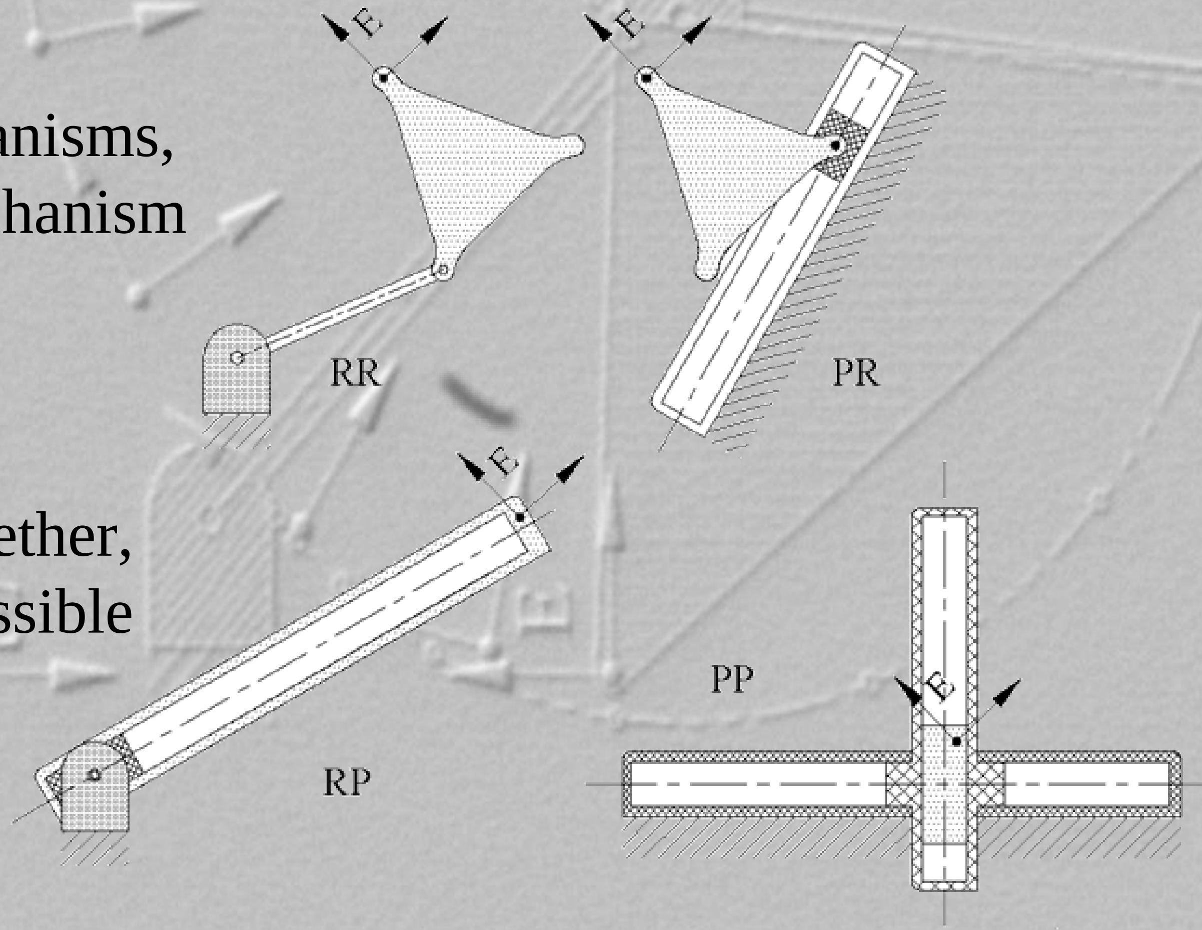
- The *five-position Burmester problem* may be stated as:
  - *given five positions of a point on a moving rigid body and the corresponding five orientations of some line on that body, design a four-bar mechanism that can move the rigid body exactly through these five poses.*



- In general, exact dimensional synthesis for rigid body guidance assumes a mechanism type (4R, slider-crank, elliptical trammel, et c.).
- Our aim is to develop an algorithm that integrates both type and approximate dimensional synthesis for *n* > 5 poses.

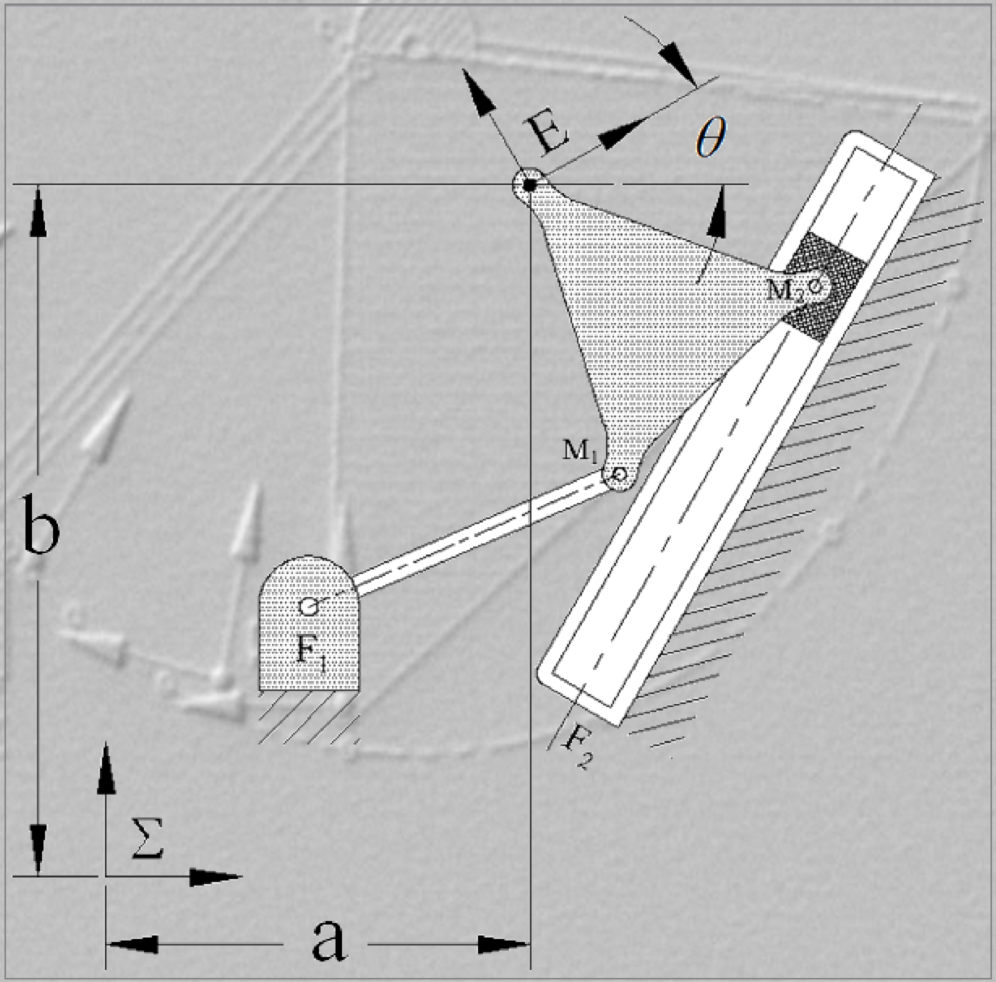
Type Synthesis

- For planar mechanisms, two types of mechanism constraints:
  - Prismatic (P);
  - Revolute (R).
- When paired together, there are four possible dyad types.



Dyad Constraints

- Dyads are connected through the coupler link at points M_1 and M_2.
  - RR – a fixed point in *E* forced to move on a fixed circle in Σ.
  - PR – a fixed point in *E* forced to move on a fixed line in Σ.
  - RP – a fixed line in *E* forced to move on a fixed point in Σ.
  - PP – a fixed line in *E* forced to move in the direction of a fixed line in Σ.

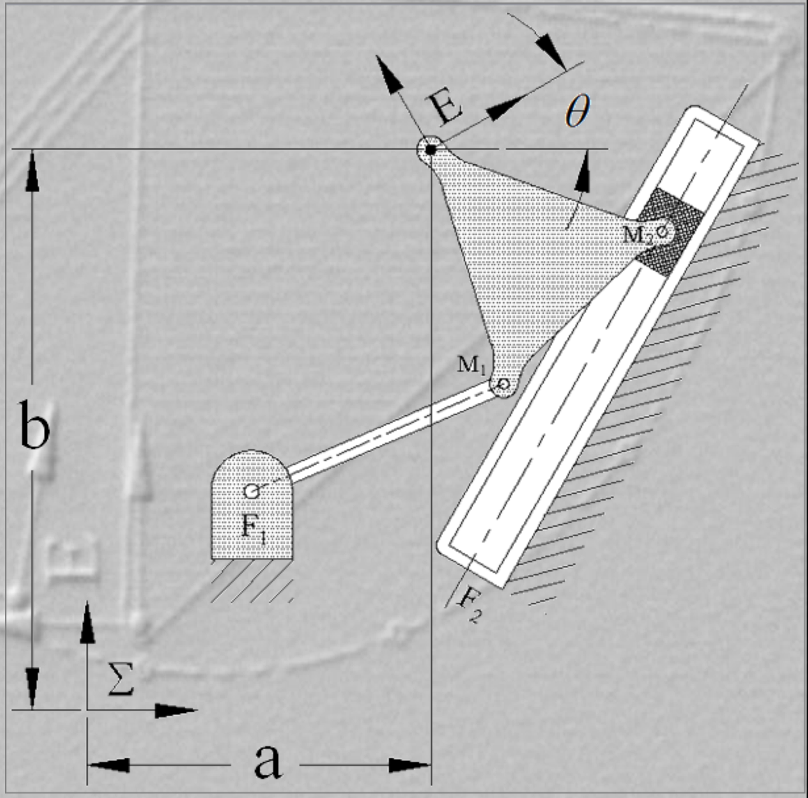


Kinematic Constraints

- Three parameters, a, b and θ describe a planar displacement of E with respect to Σ.
- The coordinates of a point in E can be mapped to those of Σ in terms of a, b and θ:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

  - $(x{:}y{:}z)$: homogeneous coordinates of a point in E.
  - $(X{:}Y{:}Z)$: homogeneous coordinates of the same point in Σ.
  - (a,b): Cartesian coordinates of O_E in Σ.
  - θ: rotation angle from X- to x-axis, positive sense CCW.



Circle and Line Coordinates

- Consider the motion of a fixed point in E constrained to move on a fixed circle in Σ, with radius r, centred on the homegeneous coordinates $(X_C : Y_C : Z)$ and having the equation

$$K_0(X^2 + Y^2) + 2K_1XZ + 2K_2YZ + K_3Z^2 = 0,$$

  where

$$K_0 = \text{arbitrary homogenising constant.}$$

  – If $K_0 = 1$, the equation represents a circle, and

$$K_1 = -X_C,$$
$$K_2 = -Y_C,$$
$$K_3 = K_1^2 + K_2^2 - r^2.$$

  – If $K_0 = 0$, the equation represents a line with line coordinates

$$[K_1 : K_2 : K_3] = \left[\frac{1}{2}L_1 : \frac{1}{2}L_2 : L_3\right].$$

Circle / Line Coordinates

Circle / Line Equation:

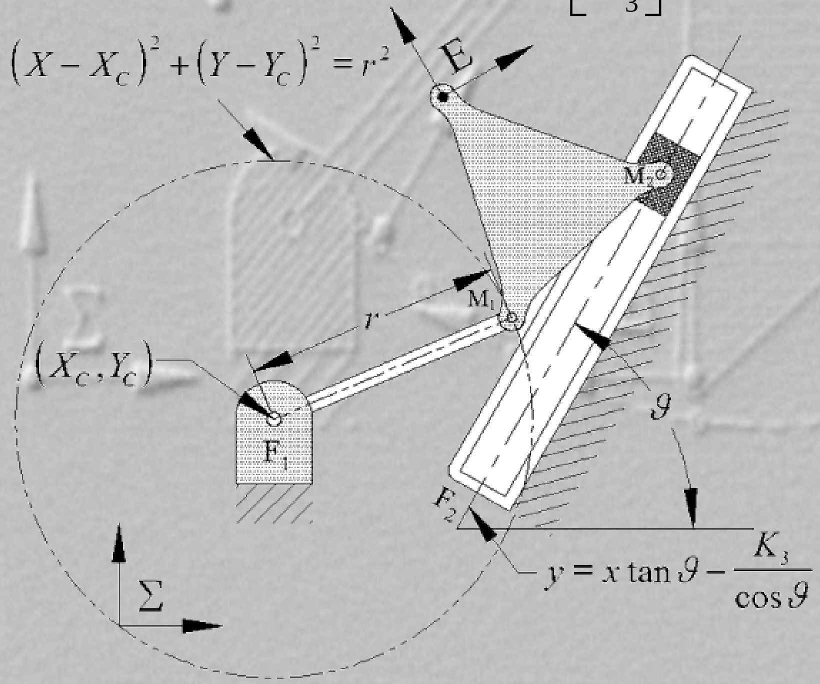
Ignoring infinitely distant coupler attachment points set $Z = z = 1$

$$\mathbf{Ck} = \begin{bmatrix} X^2 + Y^2 & 2X & 2Y & 1 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0$$

K_0 acts as a binary switch between circle and line coordinates

Circle Coordinates

$$K_0 = 1$$
$$K_1 = -X_C$$
$$K_2 = -Y_C$$
$$K_3 = K_1^2 + K_2^2 - r^2$$



Line Coordinates

$$K_0 = 0$$
$$K_1 = -\frac{\sin\vartheta_\Sigma}{2}$$
$$K_2 = \frac{\cos\vartheta_\Sigma}{2}$$
$$K_3 = x\sin\vartheta_\Sigma - y\cos\vartheta_\Sigma$$

Reference Frame Correlation

Applying $\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ to $\mathbf{Ck} = \begin{bmatrix} X^2+Y^2 & 2X & 2Y & 1 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0$

yields

$$\mathbf{Ck} = \begin{bmatrix} (x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a})^2 + (x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b})^2 \\ 2(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}) \\ 2(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}) \\ 1 \end{bmatrix}^T \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \mathbf{0}$$

- Prescribing $n > 5$ poses makes **C** an n x 4 matrix.
- **a**, **b**, and **θ** are the specified poses of *E* described in Σ.

Approximate Synthesis for n Points

For *n* poses:

$$\mathbf{Ck} = \begin{bmatrix} (x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a})^2 + (x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b})^2 \\ 2(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}) \\ 2(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}) \\ 1 \end{bmatrix}^T \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \mathbf{0}$$

- The only two unknowns in **C** are the coordinates *x* and *y* of the coupler attachment points expressed in *E*.
- For non-trivial **k** to exist satisfying **Ck=0**, then **C** must be rank deficient.
- The task is to find values for *x* and *y* that render **C** the most ill-conditioned.

Matrix Conditioning

The condition number of a matrix is defined to be:

$$\kappa \equiv \frac{\sigma_{MAX}}{\sigma_{MIN}}, \; 1 \leq \kappa \leq \infty$$

A more convenient representation is:

$$\gamma \equiv \frac{1}{\kappa}, \; 0 \leq \kappa \leq 1$$

γ is bounded both from above and below.

Choose *x* and *y* in matrix **C** such that γ is minimized.

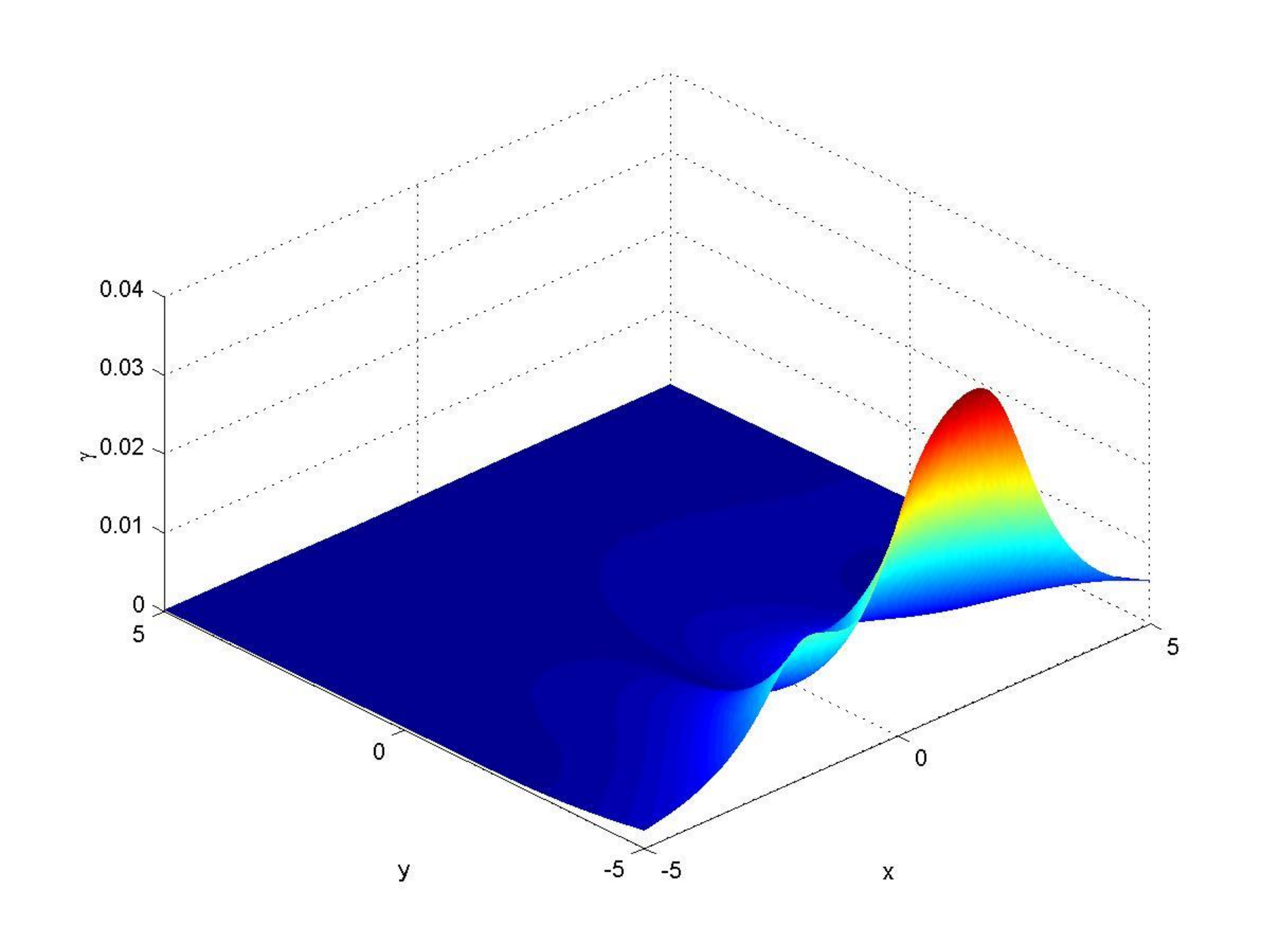
Nelder-Mead Multidimensional Simplex

- Any optimization method may be used and the numerical efficiency of the synthesis algorithm will depend on the method employed.
- We have selected the Nelder-Mead Downhill Simplex Method in Multidimensions.
- Nelder-Mead only requires function evaluations, not derivatives.
- It is relatively inefficient in terms of the required evaluations, but for this problem the computational burden is small.
- Convergence properties are irrelevant since any optimization may be used in the synthesis algorithm.
- The output of the minimization are the values of x and y that minimize the γ of **C**.

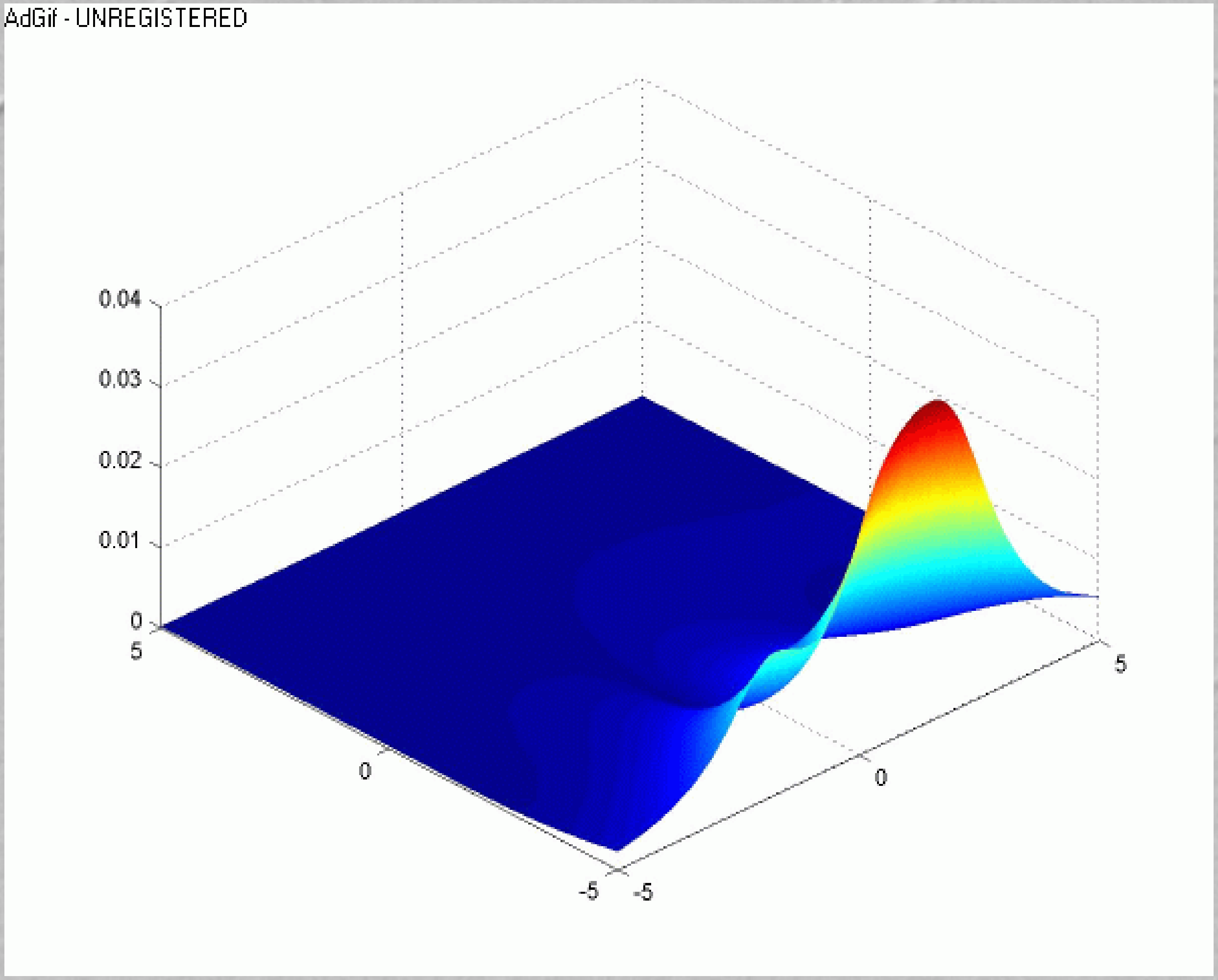
Nelder-Mead Multidimensional Simplex

- The Nelder-Mead algorithm requires an initial guess for x and y.
- We plot γ in terms of x and y in the area of $(x,y) = (0,0)$ up to the maximum distance the coupler attachments are permitted to be relative to moving coupler frame E.
- Within the corresponding parameter space, the approximate local minima are located.
- The two pairs of (x,y) corresponding to the approximate local minimum values of γ are used as initial guesses.
- The Nelder-Mead algorithm converges to the pair of (x,y) coupler attachment point locations that minimize γ within the region of interest.

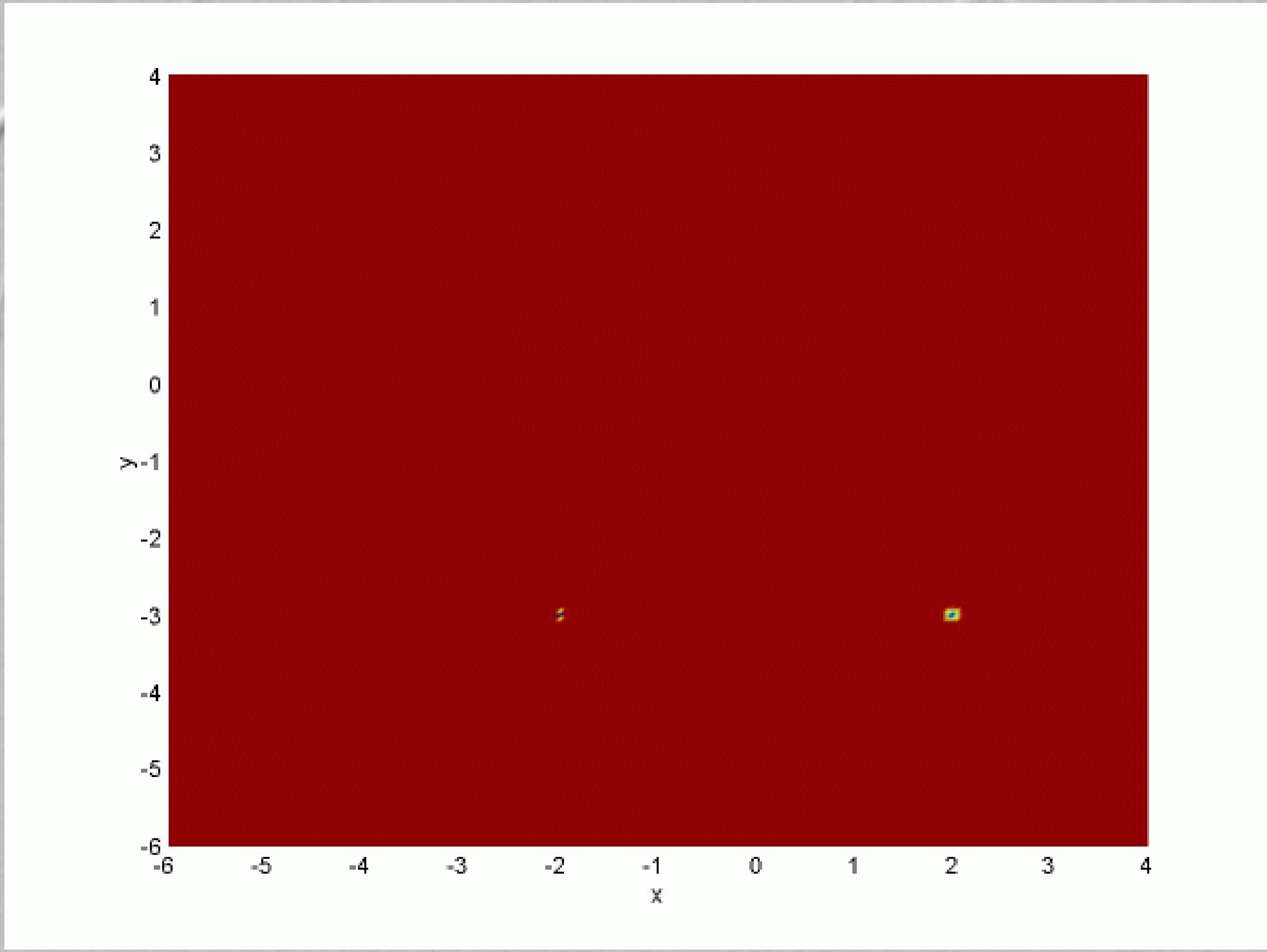
γ Plot



γ Plot



γ Plot



Nelder-Mead Minimization

- Once approximate minima are found graphically, they are input as initial guesses into the Nelder-Mead polytope algorithm
- The output of the minimization is the value of x and y that minimize the γ of **C**

Singular Value Decomposition

Any *m* x *n* matrix can be decomposed into:

$$C_{m\times n} = U_{m\times m} S_{m\times n} V^{T}_{n\times n}$$

where:

- **U** spans the range of **C**
- **V** spans the nullspace of **C**
- **S** contains the singular values of **C**

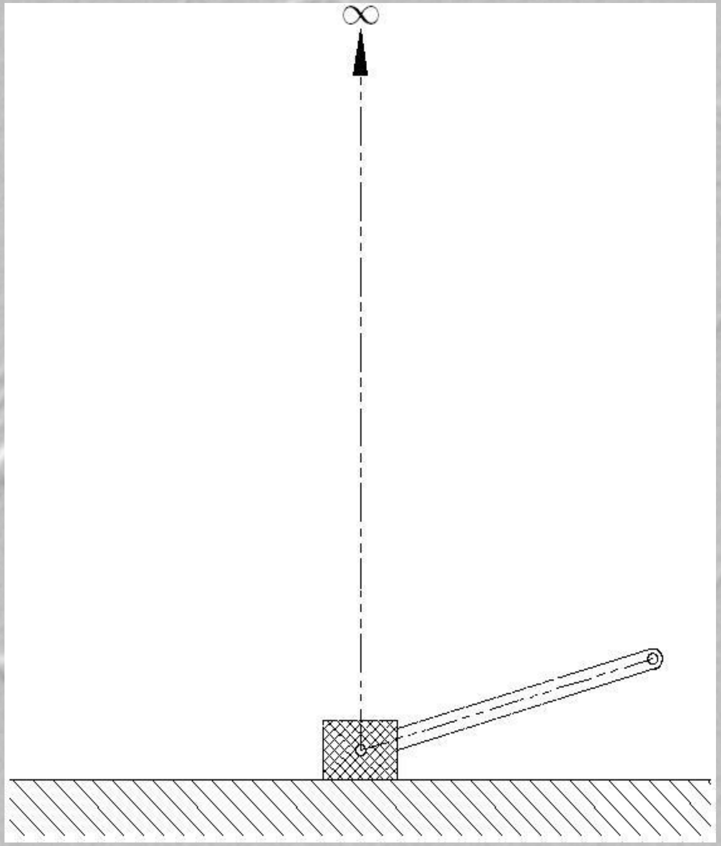
For **C** ill-conditioned (γ minimized):

- The last singular value in **S** is approximately zero
- The last column of **V** is the approximate solution to **CK** = 0

The last column of **V** is then the solution to vector **K**, defining a circle or line

Circle or Line?

- In the most general case, the vector *K* defines a circle, corresponding to an RR dyad
- If the determined circle has dimensions several orders of magnitude greater than the range of the poses, the geometry is recalculated as a line, corresponding to a PR dyad



A PR dyad, analogous to an RR dyad with infinite link length and centered at infinity

Special Cases: The RP Dyad

- RP dyads are the kinematic inverses of PR dyads
- To solve:
  - switch the roles of fixed frame Σ and moving frame *E*
  - Express points *x* and *y* in terms of *X*, *Y*, and θ
  - Solve for constant coordinates (*X*,*Y*) that minimize γ of **C**

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & -b\sin\theta - a\cos\theta \\ -\sin\theta & \cos\theta & b\cos\theta + a\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

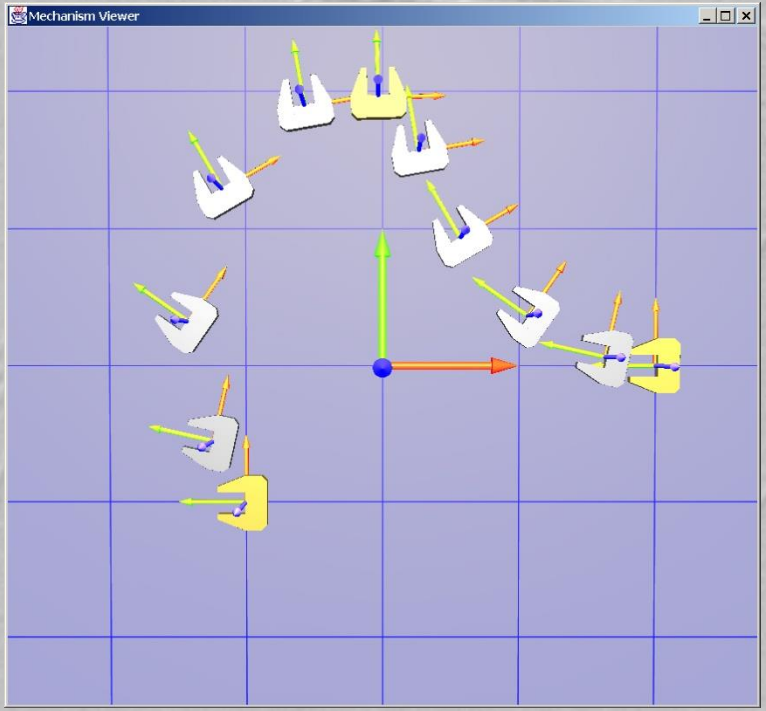
Special Cases: The PP Dyad

PP dyads:

- can *only* produce rectilinear motion at a constant orientation
- can produce *any* rectilinear motion at constant orientation
- are designed based on the practical constraints of the application

Examples: The McCarthy Design Challenge

- Issued at the ASME DETC Conference in 2002
- No information given on the mechanism used to the generate poses
- 11 poses: overconstrained problem



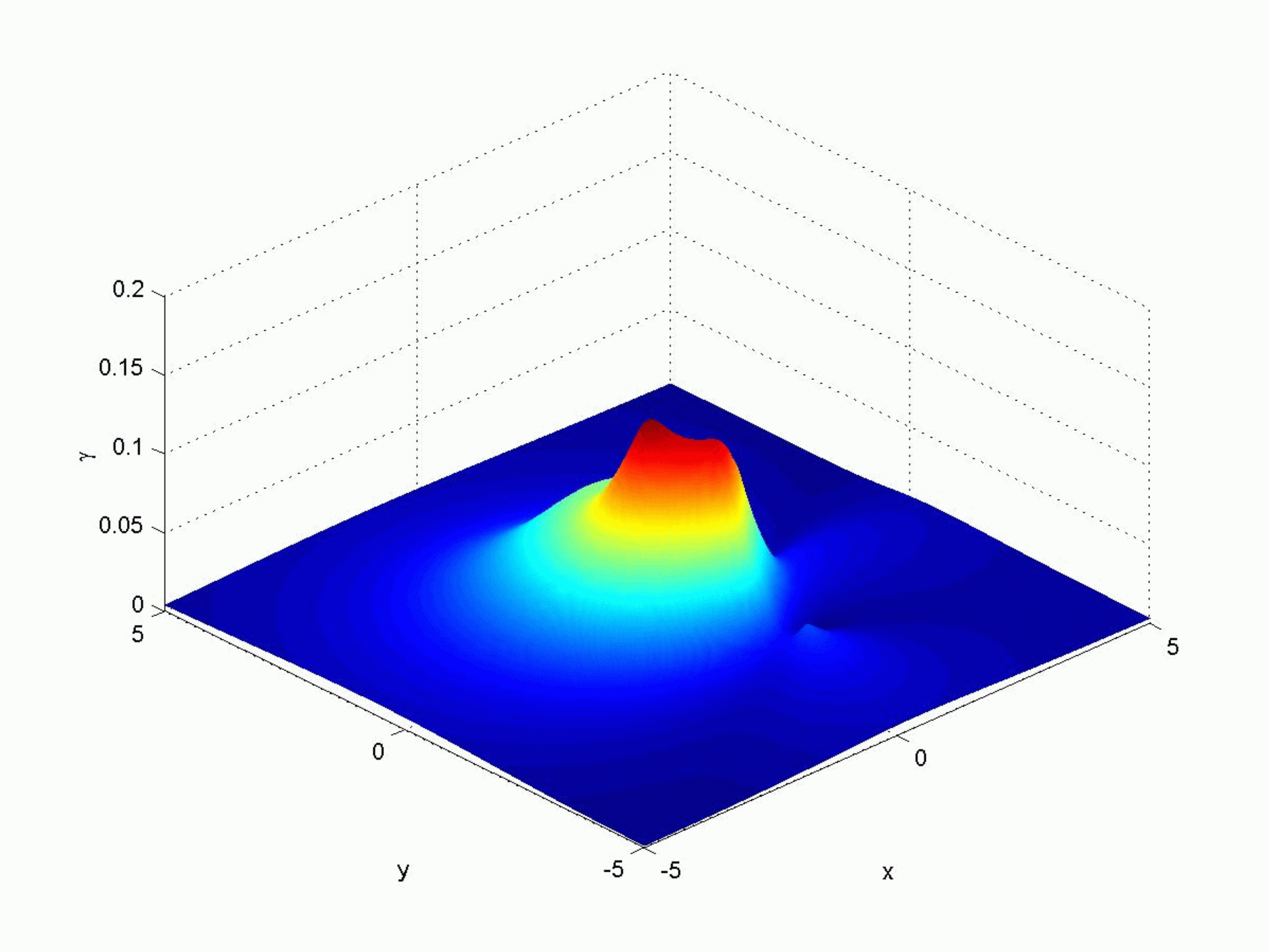
Manufacture Synthesis Matrix

- Substitute pose information into

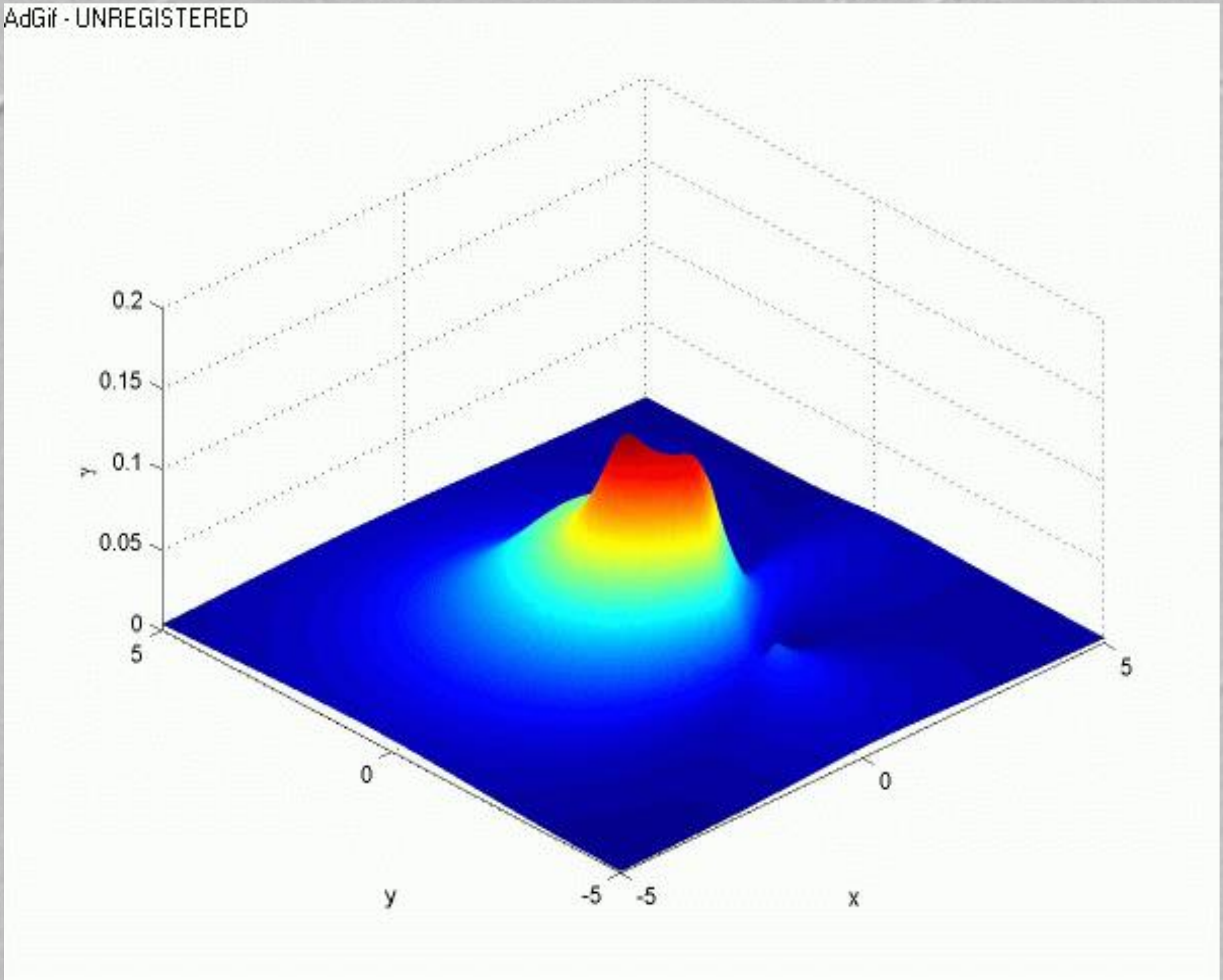
$$\mathbf{CK} = \begin{bmatrix} \left[\left(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}\right)^2 + \left(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}\right)^2 \right]^T \\ \left[2\left(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a}\right)\right] \\ \left[2\left(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b}\right)\right] \\ [1] \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = [\mathbf{0}]_{n\times 1}$$

- Plot γ in terms of x and y

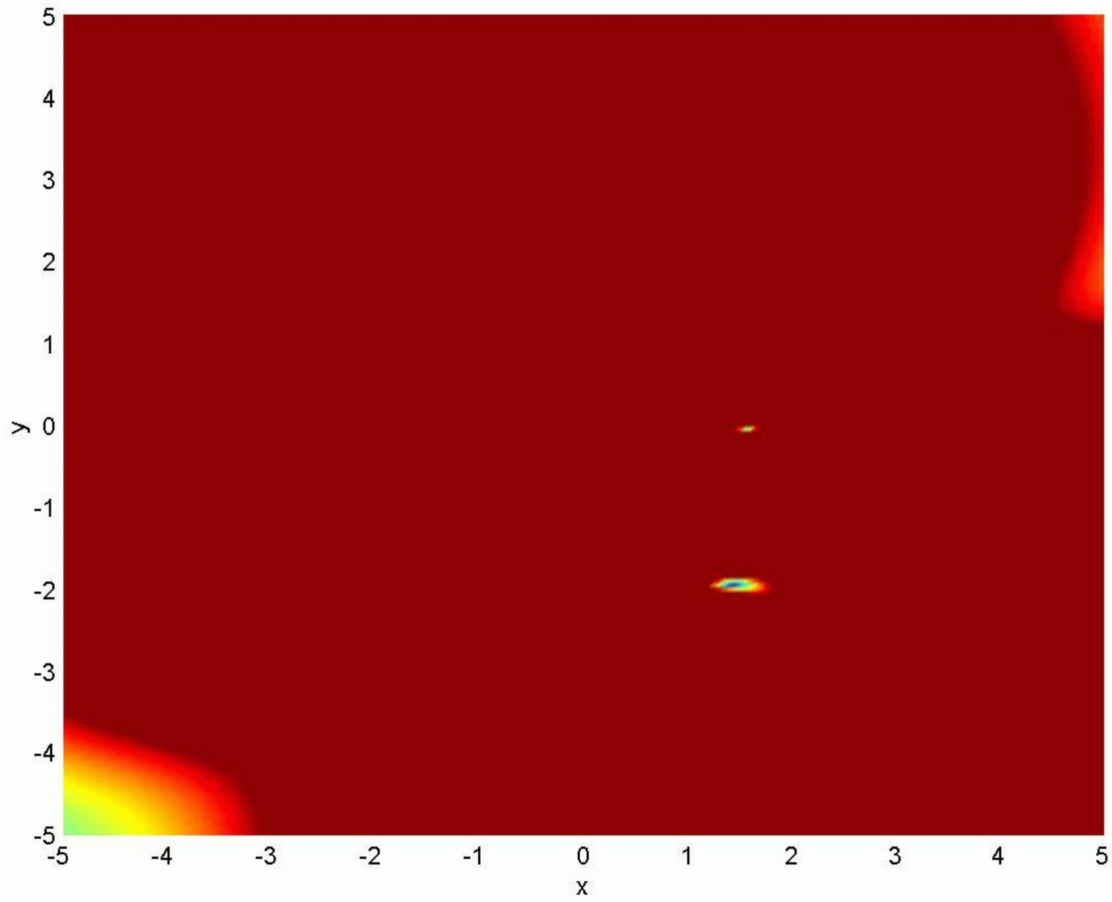
γ Plot



γ Plot



γ Plot



Extracting Mechanism Parameters

- Minima found graphically at approximately (1.5, 0.6), and (1.4, -2.0)
- Using these values as input, Nelder-Mead minimization finds the minima at

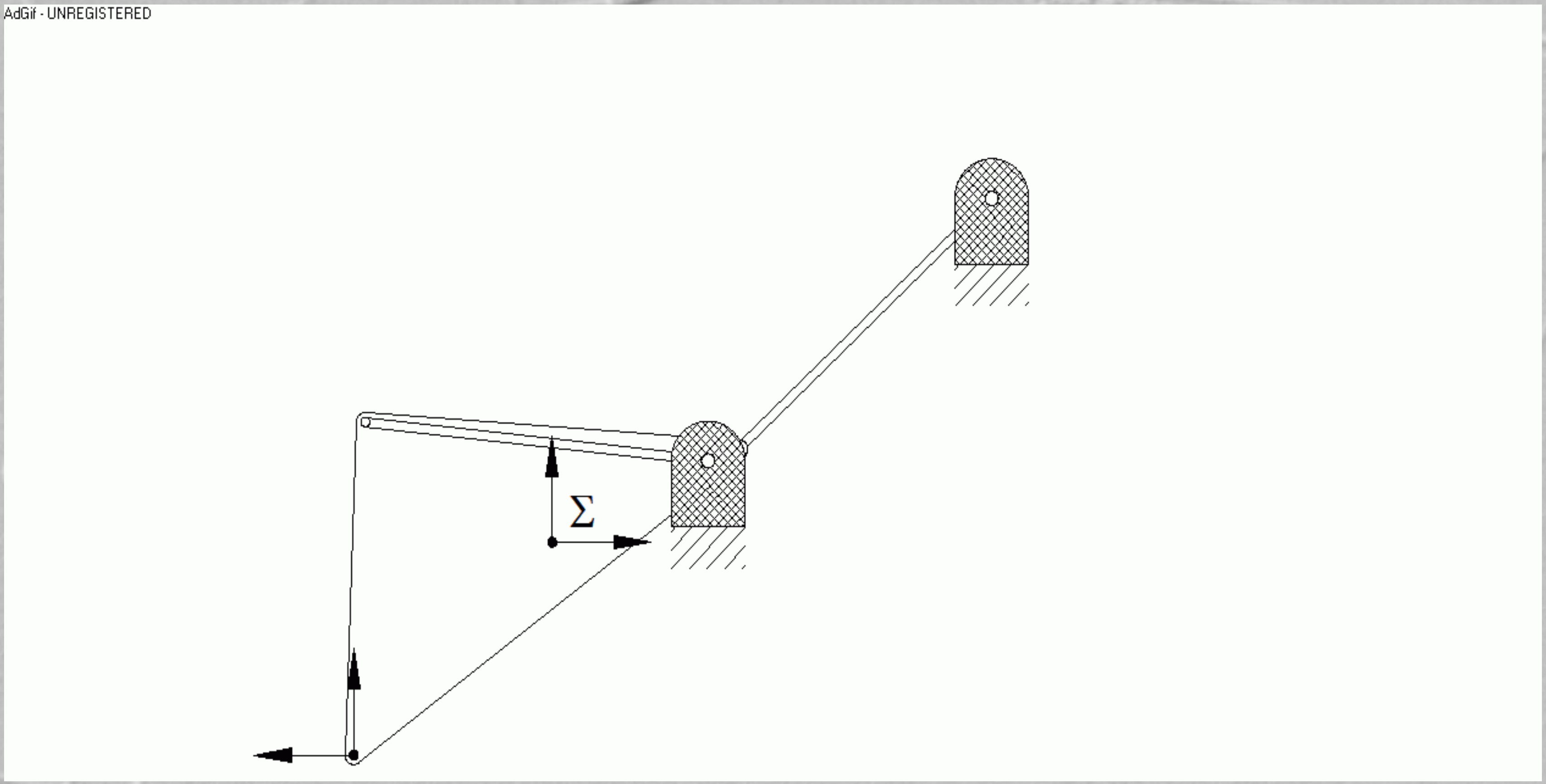
(1.5656, -0.0583) and (1.4371, -1.9415)

- Singular value decomposition is used to find the *K* vector corresponding to these coordinates

Results

	Dyad 1	Dyad 2
x	1.5656	1.4371
y	-0.0583	-1.9415
K_0	1.0000	1.0000
K_1	-0.7860	-2.2153
K_2	-0.3826	-1.6159
K_3	-2.2390	4.5236

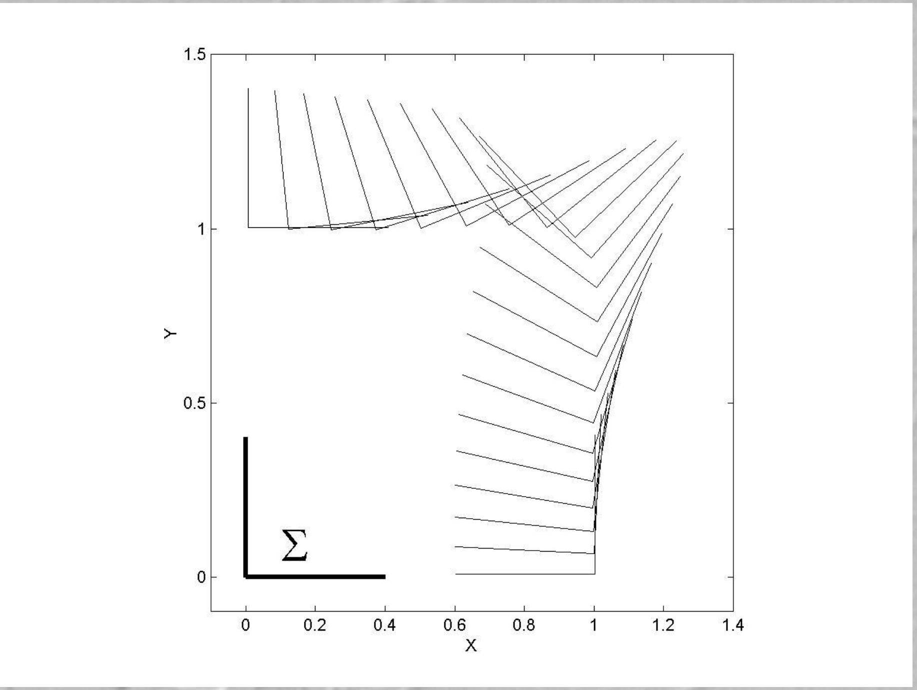
The Solution



Examples: The Square Corner

Exact synthesis is impossible for planar four-bar:

- A PPPP mechanism can replicate the positions, but not the orientations
- The coupler curve of a planar four-bar is at most 6, while a square corner requires infinite order



- Motion from (0,1) to (1,1) to (1,0)
- Orientation decreases linearly from 90 to 0 degrees

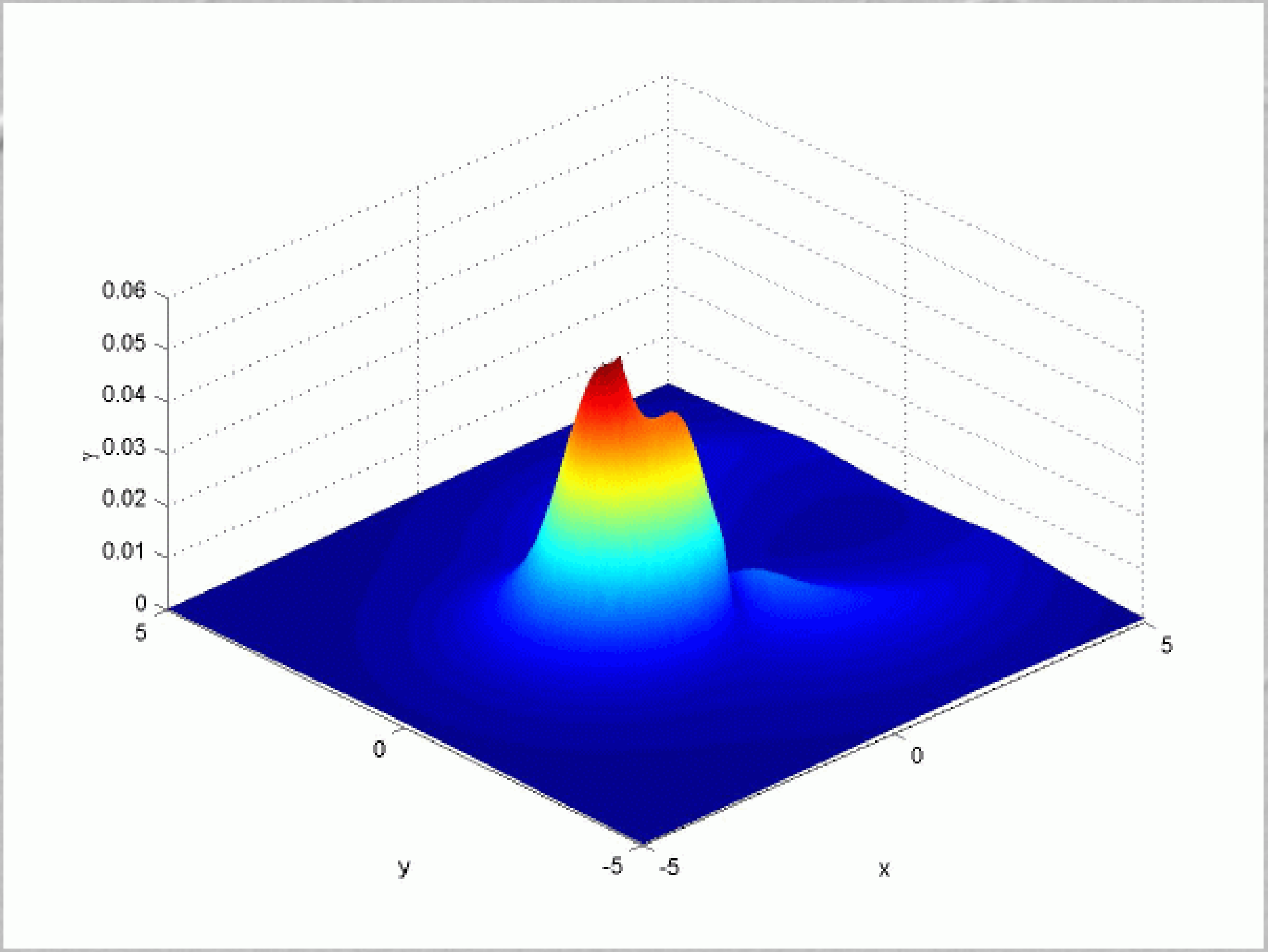
Manufacture Synthesis Matrix

- Substitute pose information into

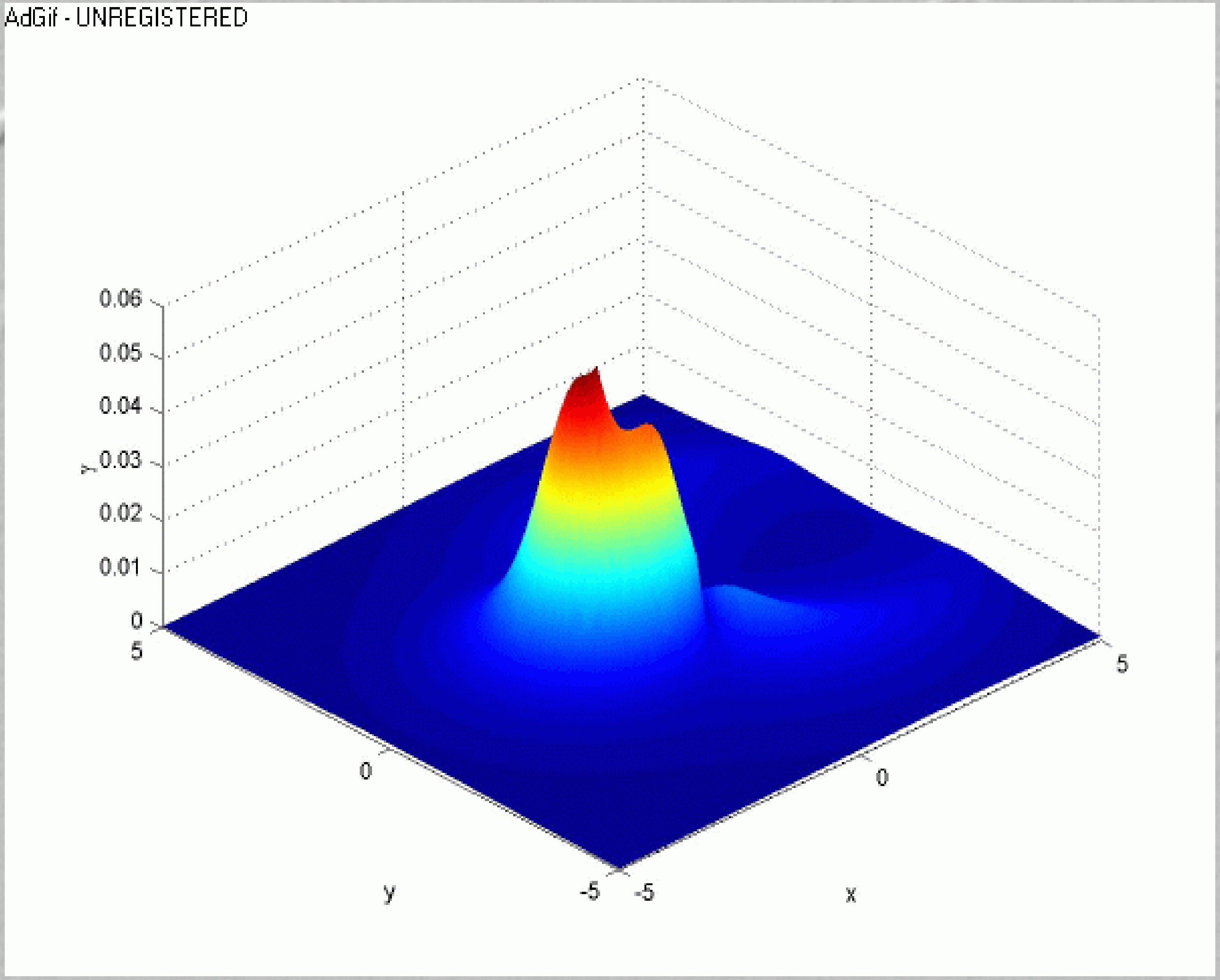
$$\mathbf{CK} = \begin{bmatrix} \left[\left(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a} \right)^2 + \left(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b} \right)^2 \right] \\ \left[2\left(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta} + \mathbf{a} \right) \right] \\ \left[2\left(x\sin\boldsymbol{\theta} - y\cos\boldsymbol{\theta} + \mathbf{b} \right) \right] \\ [1] \end{bmatrix}^T \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = [\mathbf{0}]_{n\times 1}$$

- Plot γ in terms of x and y

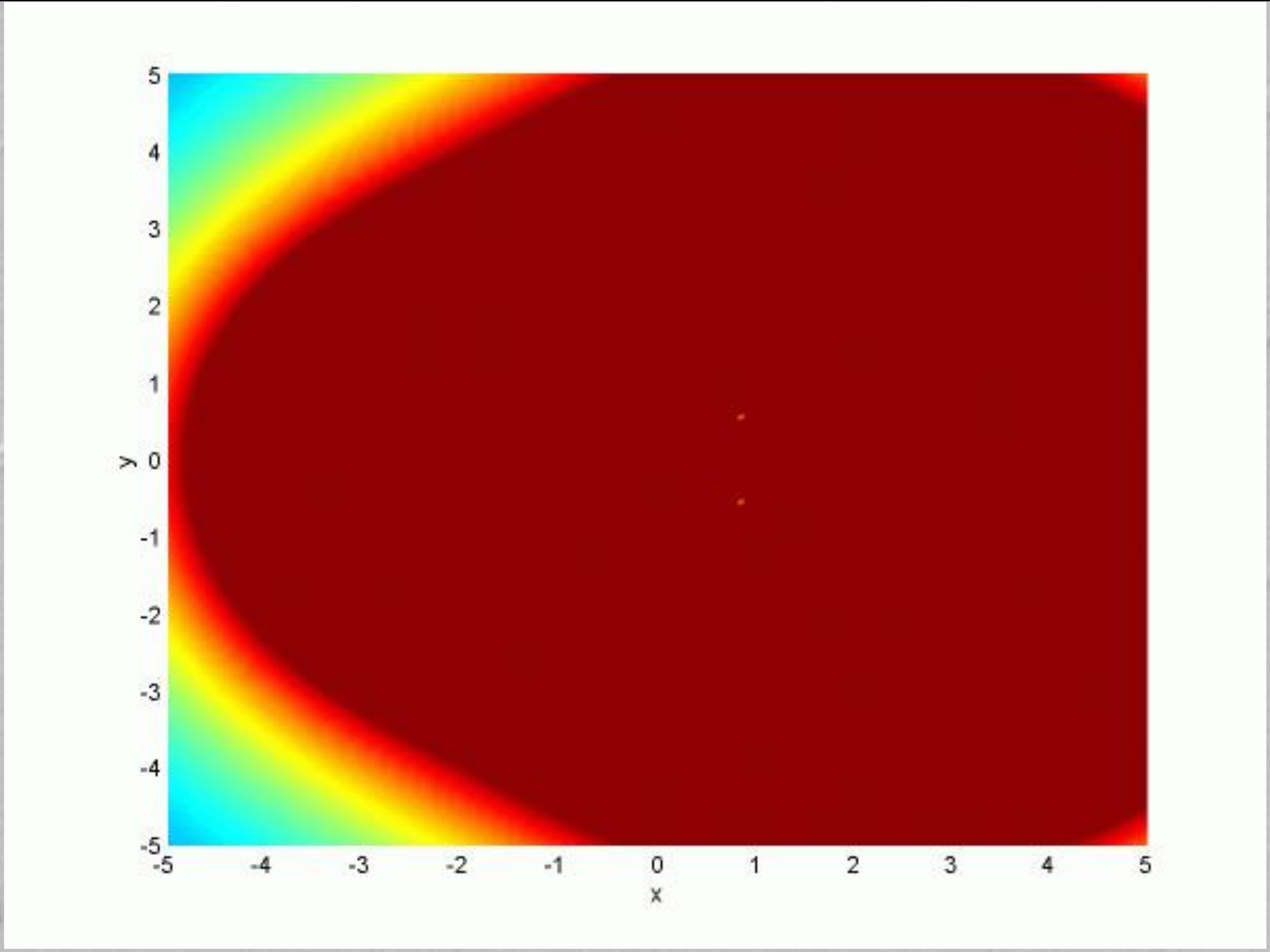
γ Plot



γ Plot



γ Plot



Extracting Mechanism Parameters

- Minima found graphically at approximately (0.8,0.6), and (0.8,-0.6)
- Using these values as input, Nelder-Mead minimization finds the minima at

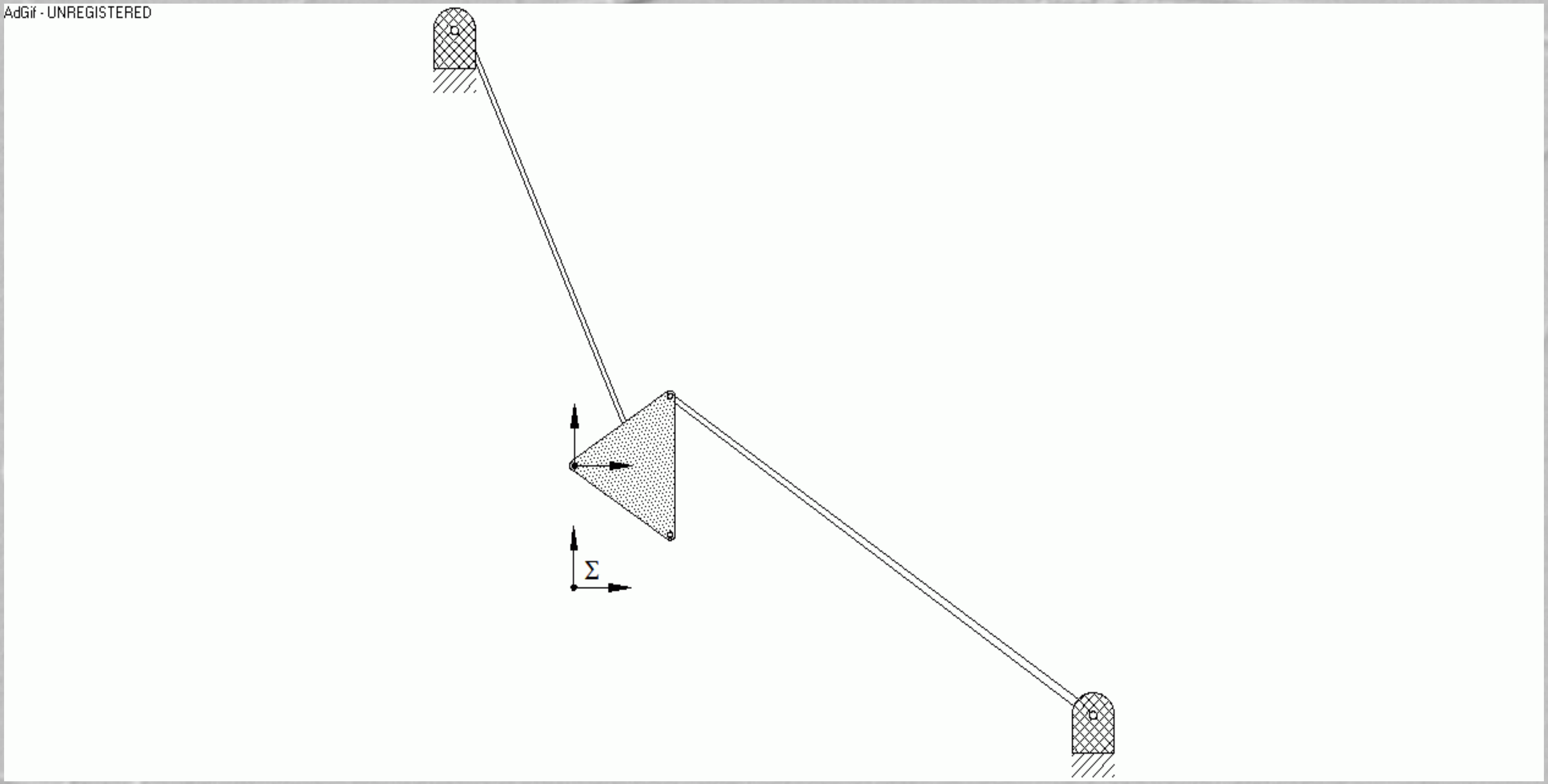
(0.8413,0.5706) and (0.8413,-0.5706)

- Singular value decomposition is used to find the *K* vector corresponding to these coordinates

Results

	Dyad 1	Dyad 2
x	0.8413	0.8413
y	0.5706	-0.5706
K_0	1.0000	1.0000
K_1	-4.5843	1.0539
K_2	1.0539	-4.5843
K_3	1.2704	1.2704

The Solution



Conclusions

- This method determines type and dimensions of mechanisms that best approximate $n > 5$ poses in a least squares sense
- No initial guess is necessary
- Examples illustrate utility and robustness

Quadric Surface Fitting Applications to Approximate Dimensional Synthesis

© M.J.D. Hayes, S. Radacina-Rusu

Department of Mechanical and Aerospace Engineering,
Carleton University, Ottawa, Canada,

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Fitting Image Space Points (Displacements) to Constraint Surfaces

- Given a suitably over constrained set of image space coordinates X_1, X_2, X_3, and X_4 which represent the desired set of positions and orientations of the coupler identify the constraint surface shape coefficients: K_0, K_1, K_2, K_3, x, and y.
- The given image space points are on some space curve.
- Project these points onto the *best* 4^{th} order curve of intersection of two quadric constraint surfaces.
- These intersecting surfaces represent two dyads in a mechanism that possesses displacement characteristics closest to the set of specified poses.

Identifying the Constraint Surfaces

- Surface type is embedded in in the coefficients of its implicit equation:

$$c_0X_4^2+c_1X_1^2+c_2X_2^2+c_3X_3^2+c_4X_1X_2+c_5X_2X_3+$$
$$c_6X_3X_1+c_7X_1X_4+c_8X_2X_4+c_9X_3X_4=0.$$

- It can be classified according to certain invariants of its discriminant and quadratic forms:

$$\begin{bmatrix}X_1\\X_2\\X_3\\X_4\end{bmatrix}^T\begin{bmatrix}c_1 & \frac{1}{2}c_4 & \frac{1}{2}c_6 & \frac{1}{2}c_7\\ \frac{1}{2}c_4 & c_2 & \frac{1}{2}c_5 & \frac{1}{2}c_8\\ \frac{1}{2}c_6 & \frac{1}{2}c_5 & c_3 & \frac{1}{2}c_9\\ \frac{1}{2}c_7 & \frac{1}{2}c_8 & \frac{1}{2}c_9 & c_0\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\\X_4\end{bmatrix}=\mathbf{X}^T\mathbf{D}\mathbf{X}=0.$$

Identifying the Constraint Surfaces

- Given a sufficiently large number *n* of poses expressed as image space coordinates yields *n* equations linear in the c_i coefficients

$$c_0X_4^2 + c_1X_1^2 + c_2X_2^2 + c_3X_3^2 + c_4X_1X_2 + c_5X_2X_3 +$$
$$c_6X_3X_1 + c_7X_1X_4 + c_8X_2X_4 + c_9X_3X_4 = 0.$$

- The n equations can be re-expressed as:

$$\mathbf{Ac} = \mathbf{0}.$$

- The same numbered elements in Matrix **A**, corresponding to the X_i are scaled by the unknown c_i.

Identifying the Constraint Surfaces

- Applying SVD to Matrix **A** reveals the vectors **c** that are in, or computationally close, in a least-squares sense, to the nullspace of **A**.
- Certain invariants of the resulting discriminant and corresponding quadratic form reveal the nature of the quadric surface.
- *RR* dyads require the quadric surface to be an hyperboloid of one sheet with certain properties.
- *RP* and *PR* dyads require the quadric surface to be an hyperbolic paraboloid.

Equivalent Minimization Problem

- Assuming the mechanism type has been identified given $n >> 5$ specified poses, the approximate synthesis problem can be solved using an equivalent unconstrained non-linear minimization problem.
- It can be stated as "find the surface shape parameters that minimize the total spacing between all points on the specified reference curve and the same number of points on a dyad constraint surface".
- First the constraint surfaces are projected into the space corresponding to the hyperplane $X_4=1$.
- This yields the following parameterizations:

Equivalent Minimization Problem

- Hyperboloid of one sheet:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} ([x-K_1]t+K_2+y)+(r\sqrt{t^2+1})\cos\gamma \\ ([y-K_2]t-K_1-x)+(r\sqrt{t^2+1})\sin\gamma \\ 2t \end{bmatrix},$$

$$\gamma \in \{0, \Lambda\ , 2\pi\},\ t \in \{-\infty, \Lambda\ , \infty\},$$

x and y are the coordinates of the moving revolute centre expressed in the moving coordinate system E,

K_1 and K_2 are the coordinates of the fixed revolute centre expressed in the fixed coordinate system Σ,

r is the distance between fixed and moving revolute centres,

while *t* and *γ* are free parameters.

Equivalent Minimization Problem

- Hyperbolic paraboloid:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \\ t \end{bmatrix} + s \begin{bmatrix} -b(t) \\ a(t) \\ 0 \end{bmatrix}$$

$$t \in \{-\infty, \Lambda, \infty\},\ s \in \{-\infty, \Lambda, \infty\},$$

  f(*t*), *g*(*t*), *a*(*t*), and *b*(*t*) are functions of the surface shape parameters and the free parameter *t*,

  while *s* is another free parameter.

- Note that in both cases the X_3 coordinate varies linearly with the free parameter *t*, and can be considered another free parameter.

Equivalent Minimization Problem

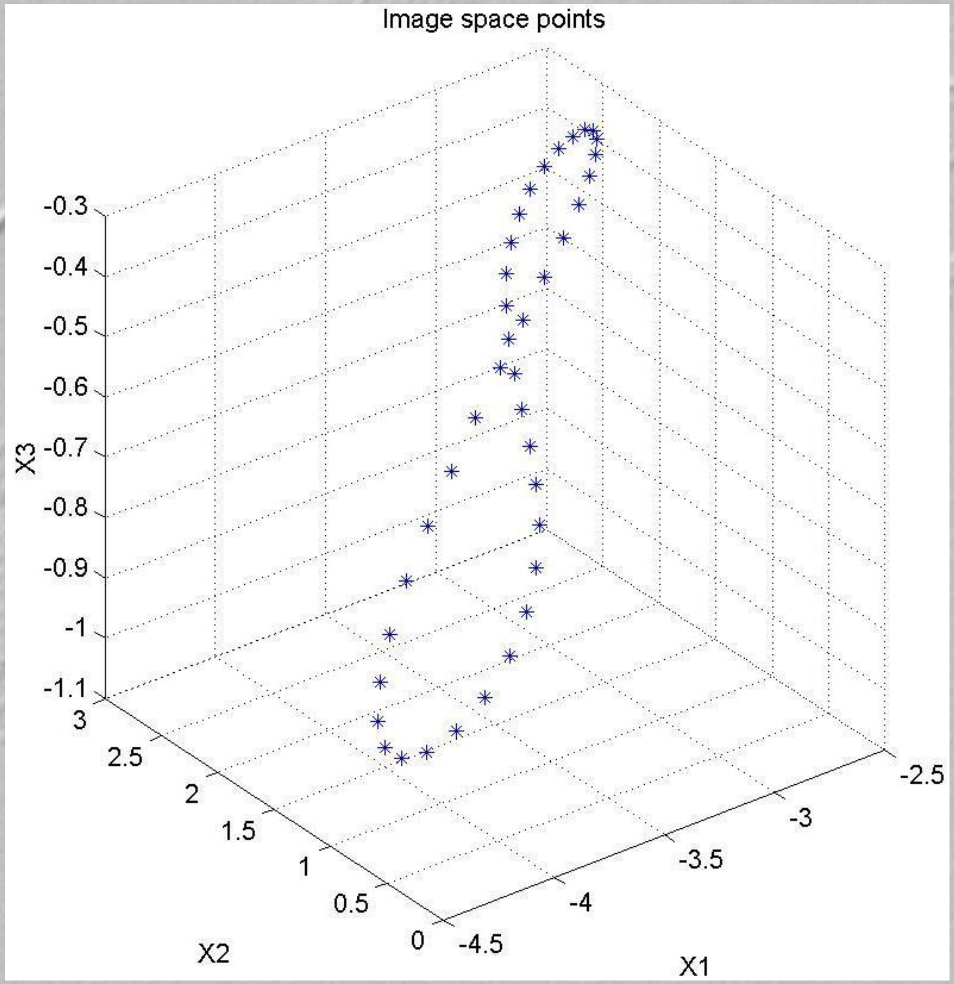
- The total distance between the specified reference image space points on the reference curve and corresponding points that lie on a constraint surface where $t=X_3=X_{3_{\mathrm{ref}}}$ is defined as

$$d=\sum_{i=1}^{n}\sqrt{(X_{1_{\mathrm{ref}}}-X_{1_i})^2+(X_{2_{\mathrm{ref}}}-X_{2_i})^2}$$

- The two sets of surface shape parameters that minimize d represent the two best constraint surfaces that intersect closest to the reference curve.
- The distance between each reference point and each corresponding point on the quadric surface in the hyperplane $t=X_3=X_{3_{\mathrm{ref}}}$ can measured in the plane spanned by X_1 and X_2.

Example

- A planar 4*R* linkage was used to generate 40 poses of the coupler.
- The resulting image space points lie on the curve of intersection of two hyperboloids of one sheet.
- The reference curve can be visualized in the hyperplane $X_4=1$.



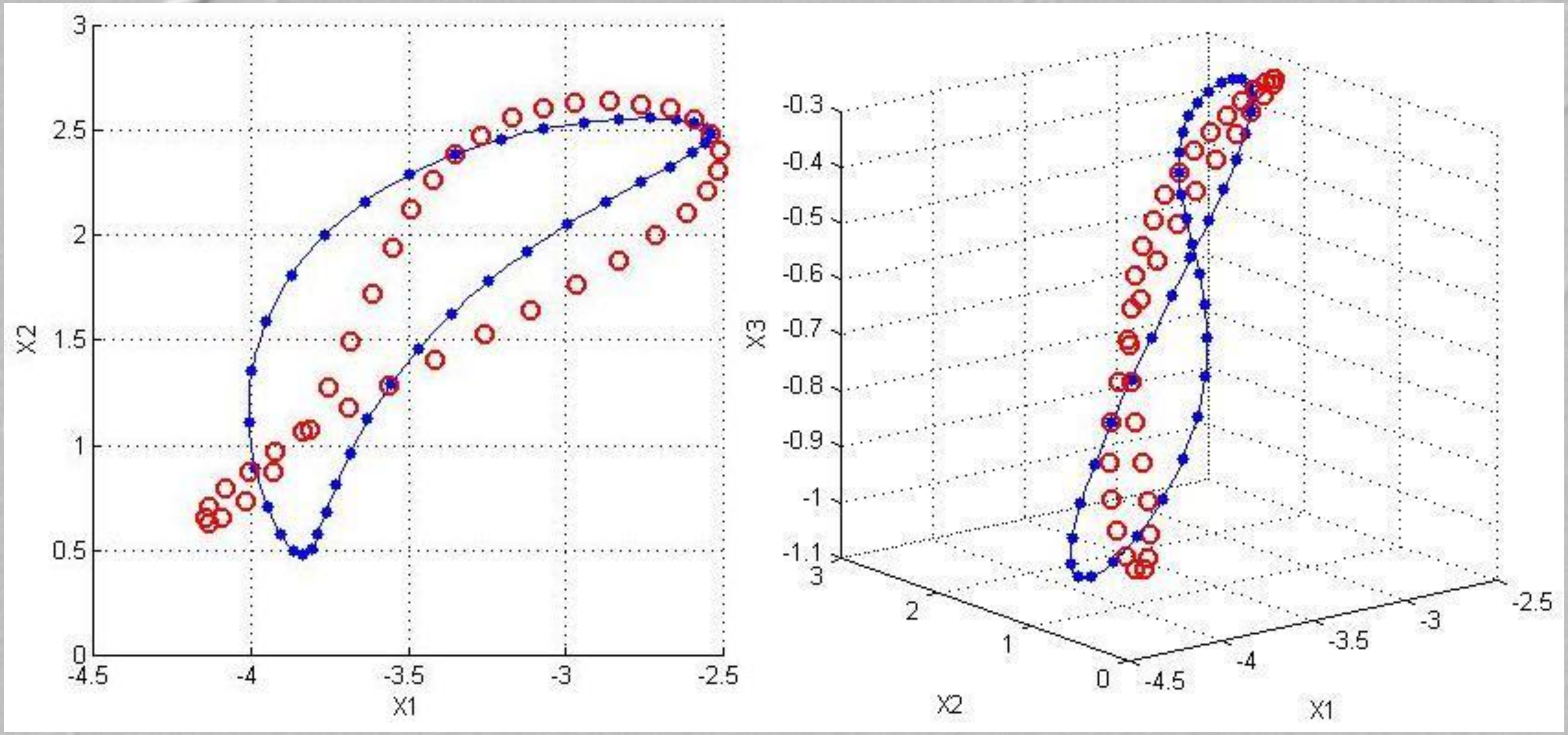
Example

- In order for the algorithm to converge to the solution that minimizes *d*, decent initial guesses for the surface shape parameters are required.
- Out of the 40 reference points, sets of five were arbitrarily chosen spaced relatively wide apart yielding sets of five equations in the five unknown shape parameters.
- Solving yields the initial guesses.
- Non-linear unconstrained algorithms such as the Nelder-Mead simplex and the Hookes-Jeeves methods were used with similar outcomes.

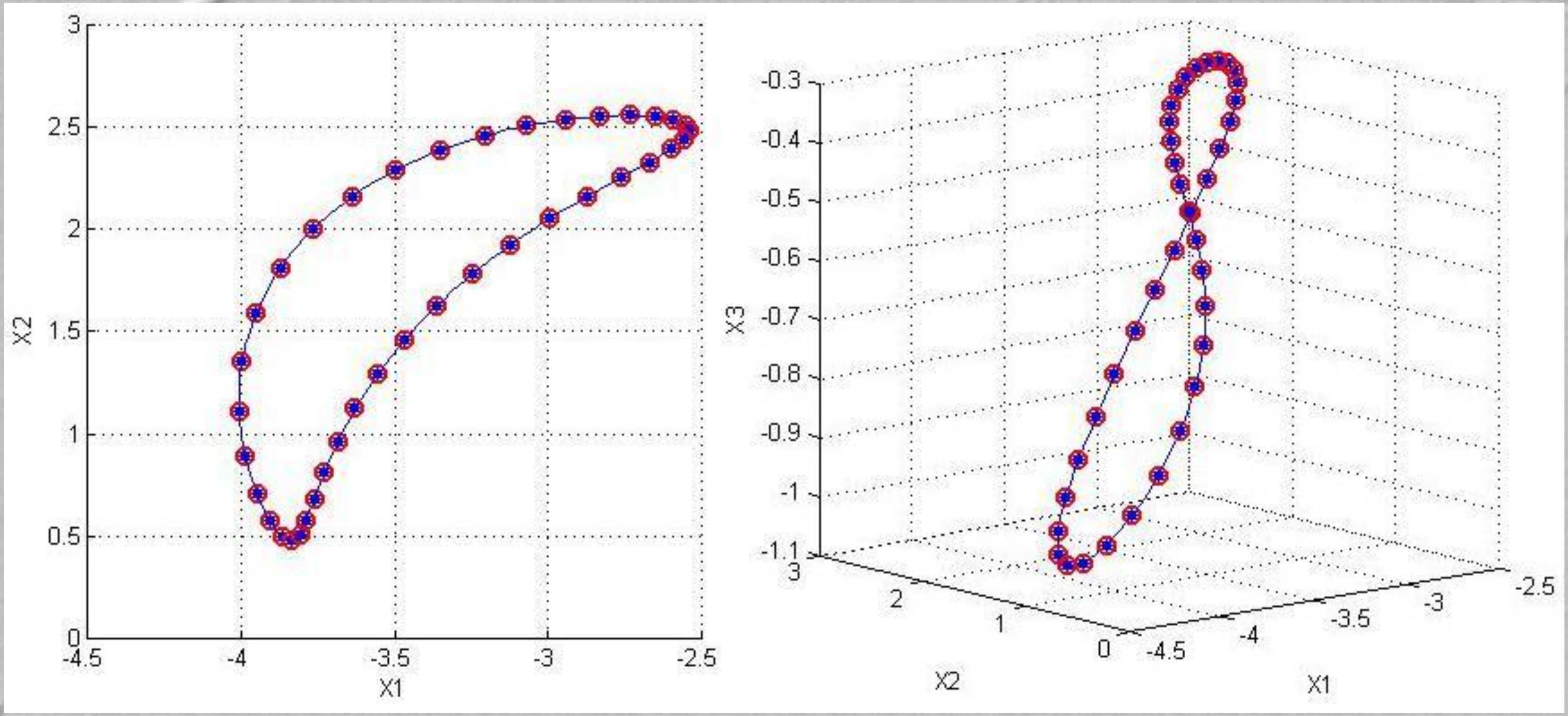
Results Generated From Initial Guesses

Parameter	Guess 1	Guess 2	Guess 3	Guess 4	Guess 5	Guess 6	Guess 7
K_1	-97.720	-18.202	888.914	-5.000	1.000	-25.445	-1.398
K_2	-57.463	-12.363	432.395	0.000	-1.000	-17.073	-6.191
K_3	1491.757	261.650	-2374.375	21.000	-23.000	390.531	36.554
x	-1.133	-1.287	-0.894	3.000	-1.000	-1.309	-4.388
y	0.534	0.889	-5.375	-2.000	-2.000	1.030	-2.361
Iterations	450	623	718	101	176	745	436
d	1.1132	1.9333	6.726	0.0004	0.0010	1.5746	4.8138

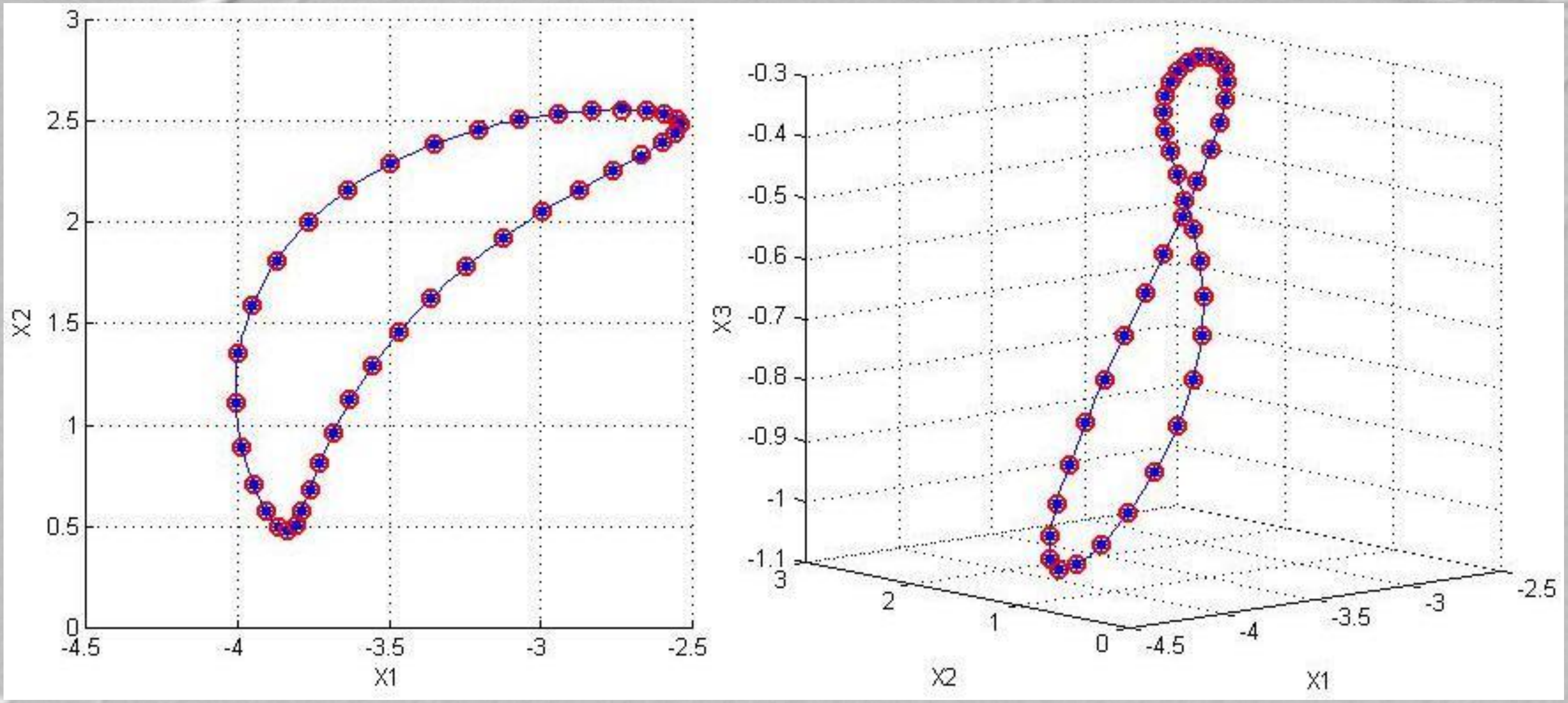
Results: Initial Guess 3



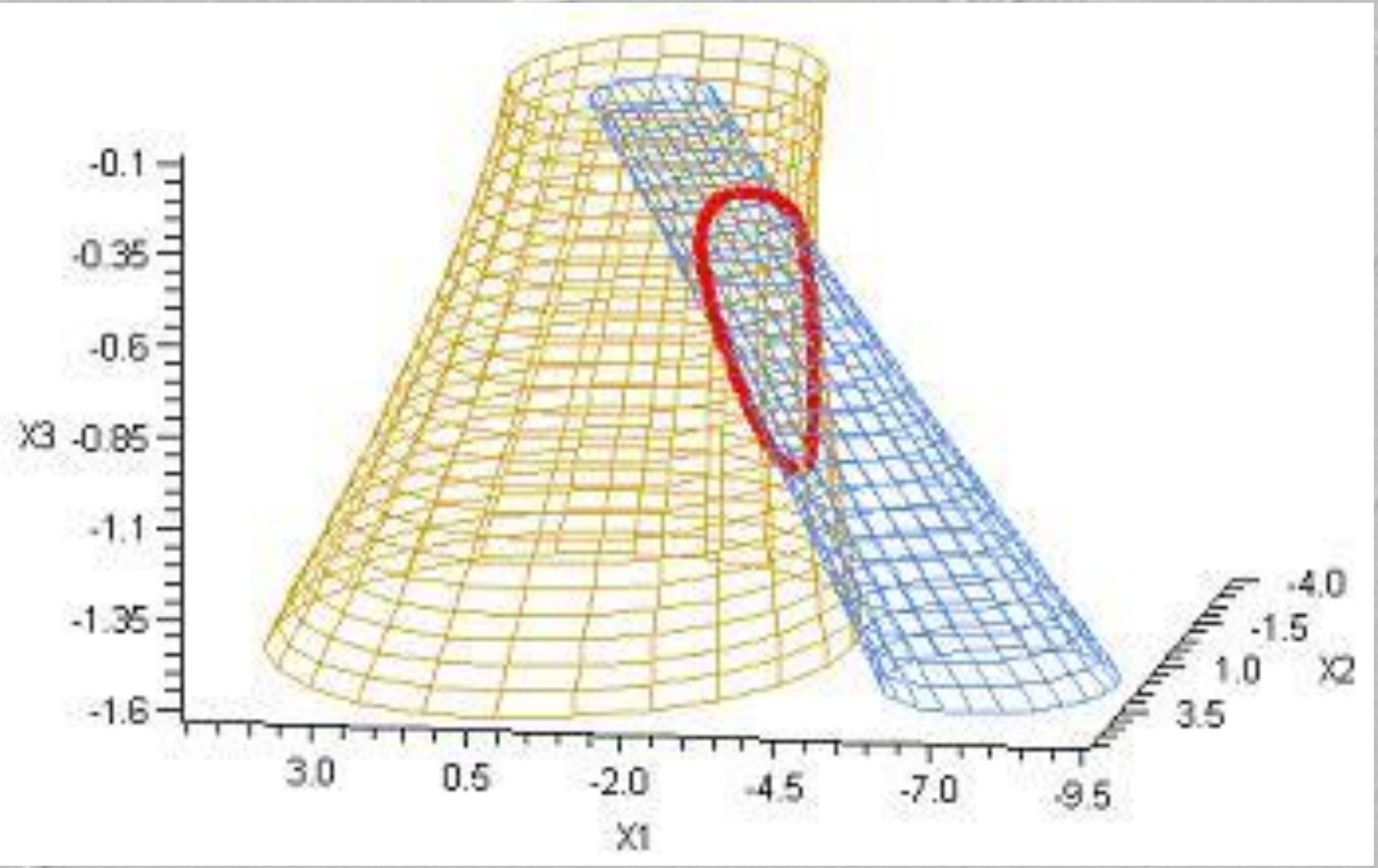
Results: Initial Guess 4



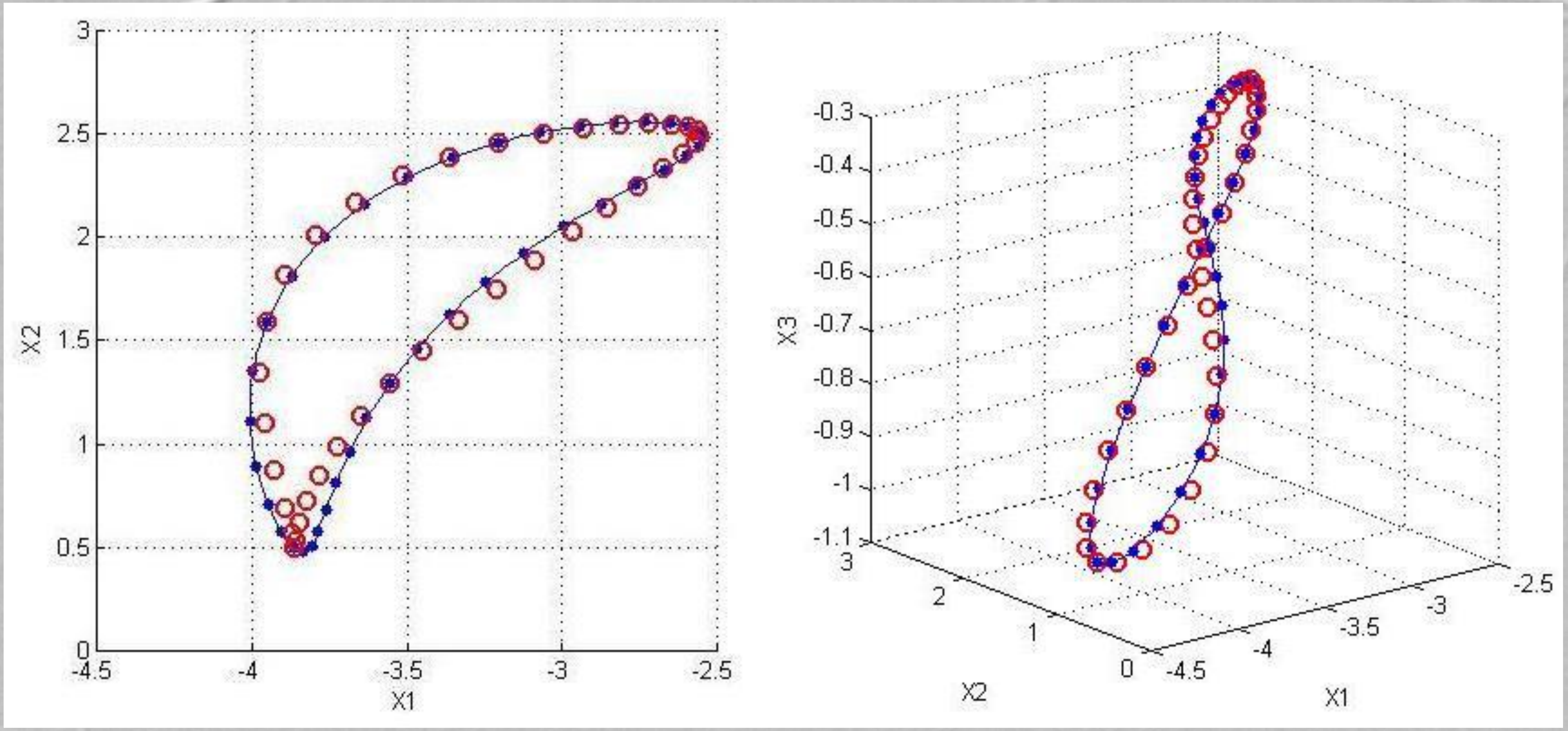
Results: Initial Guess 5



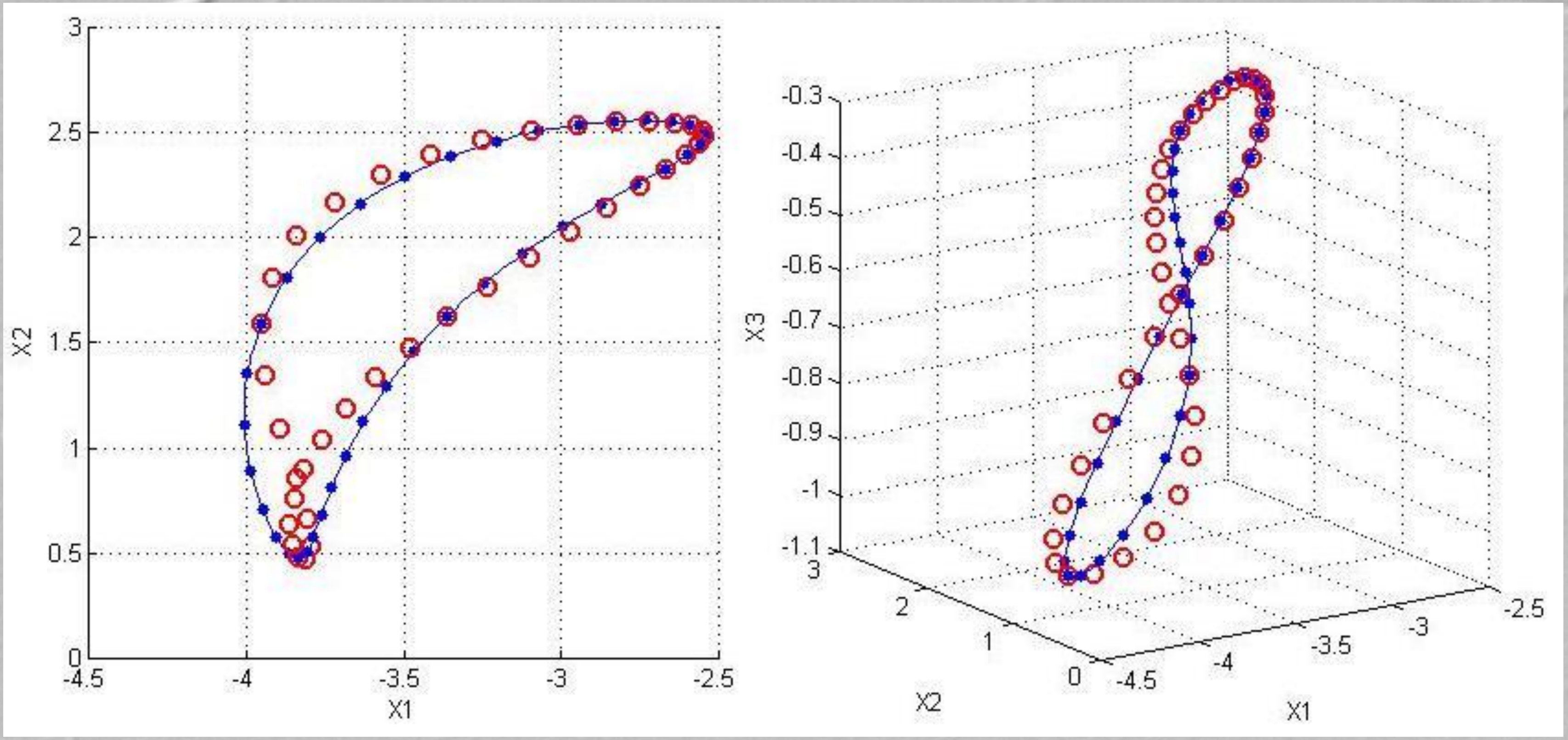
Intersection of Two *Best* Surfaces



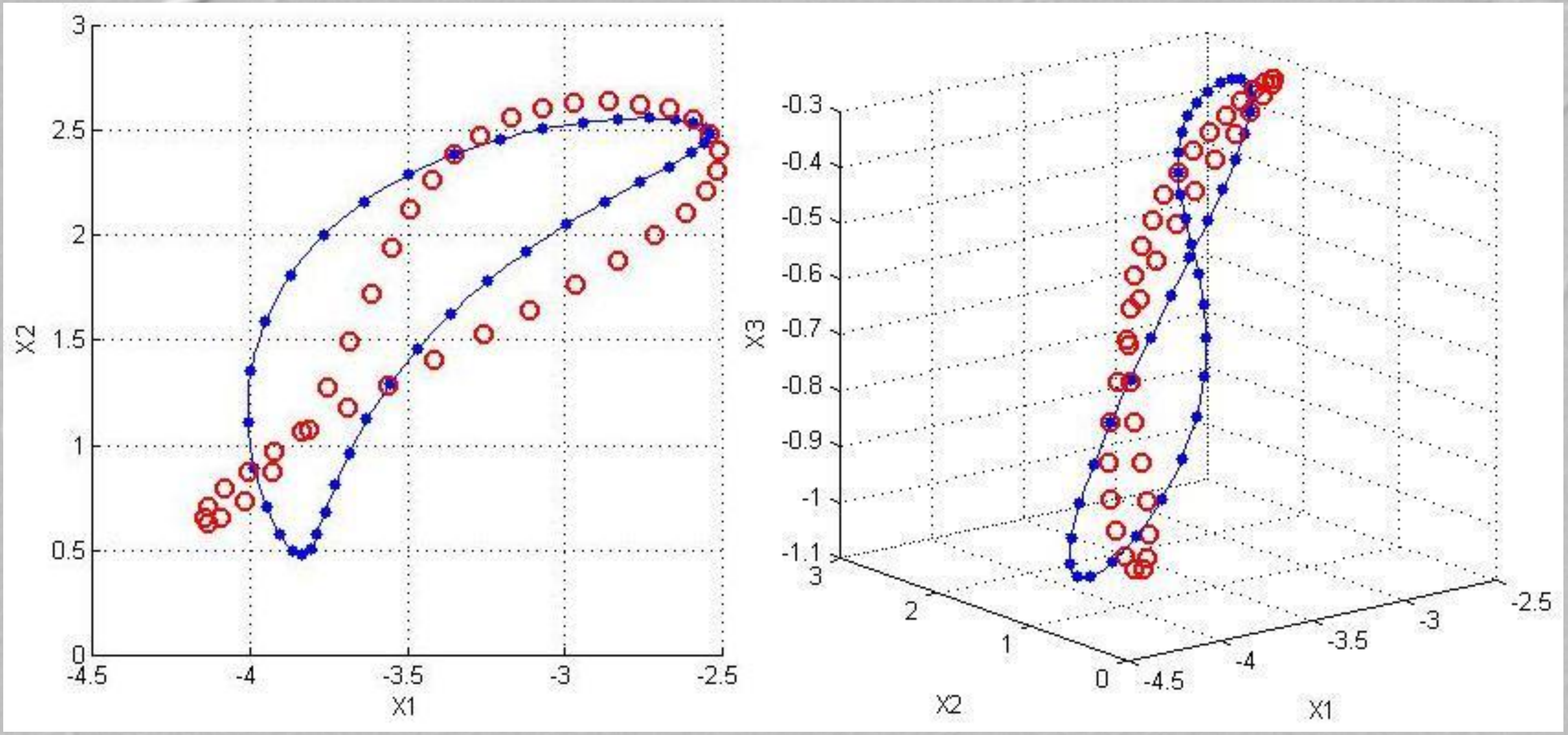
Results: Initial Guess 1



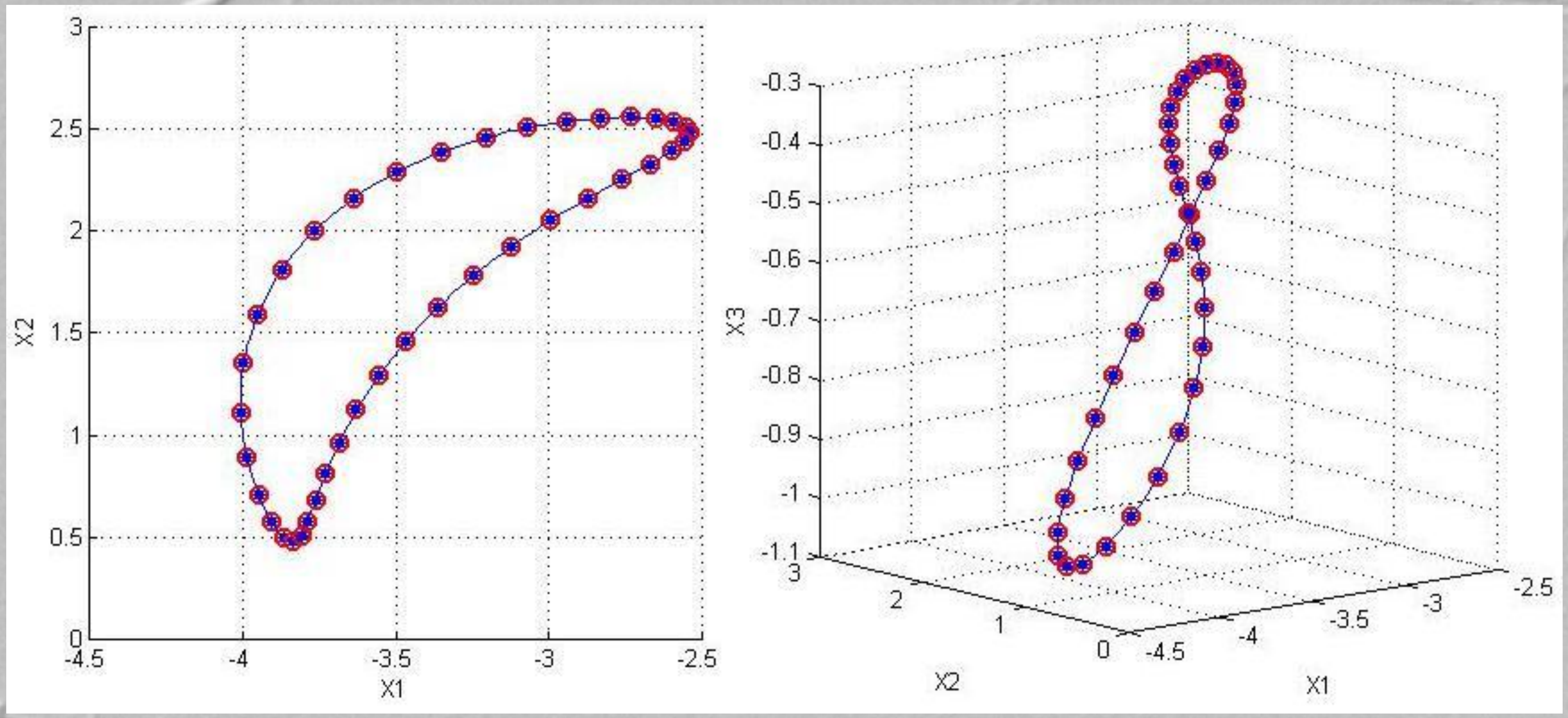
Results: Initial Guess 2



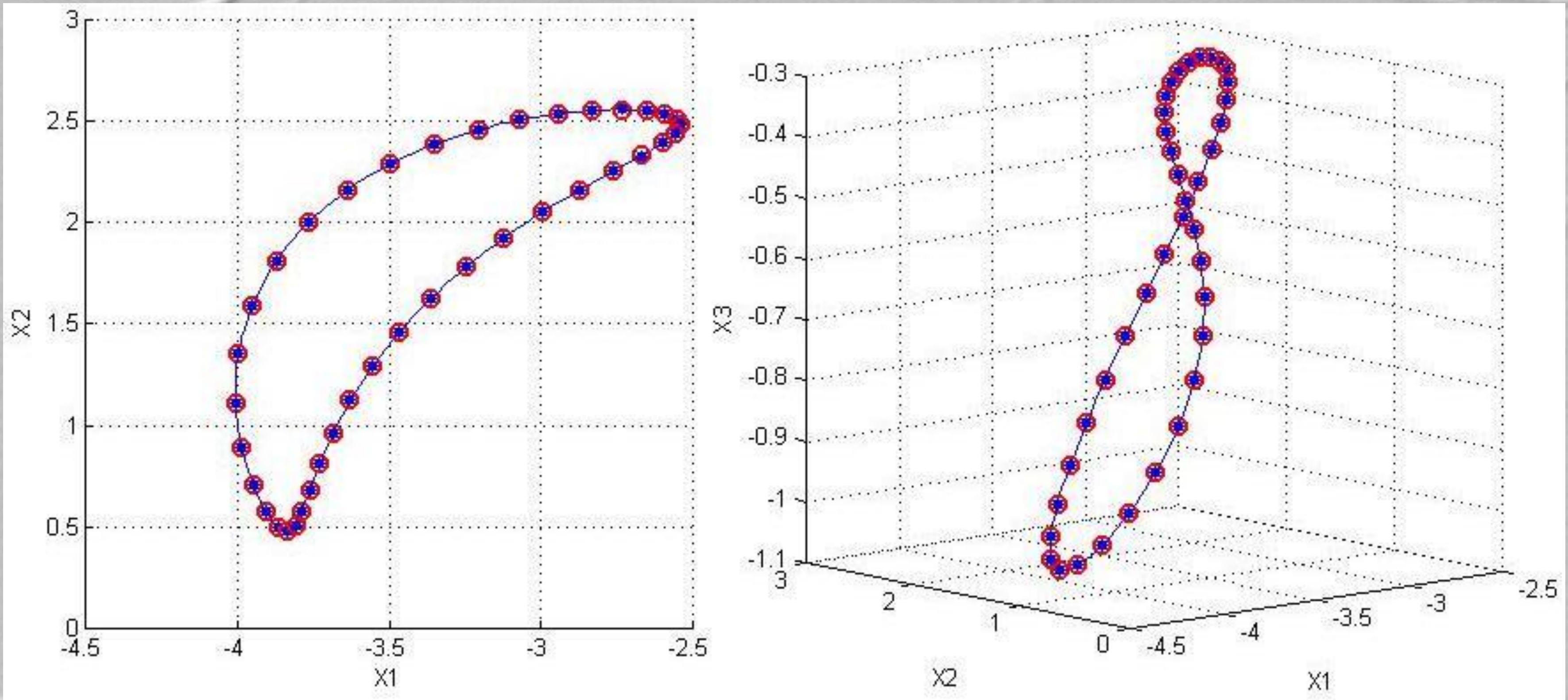
Results: Initial Guess 3



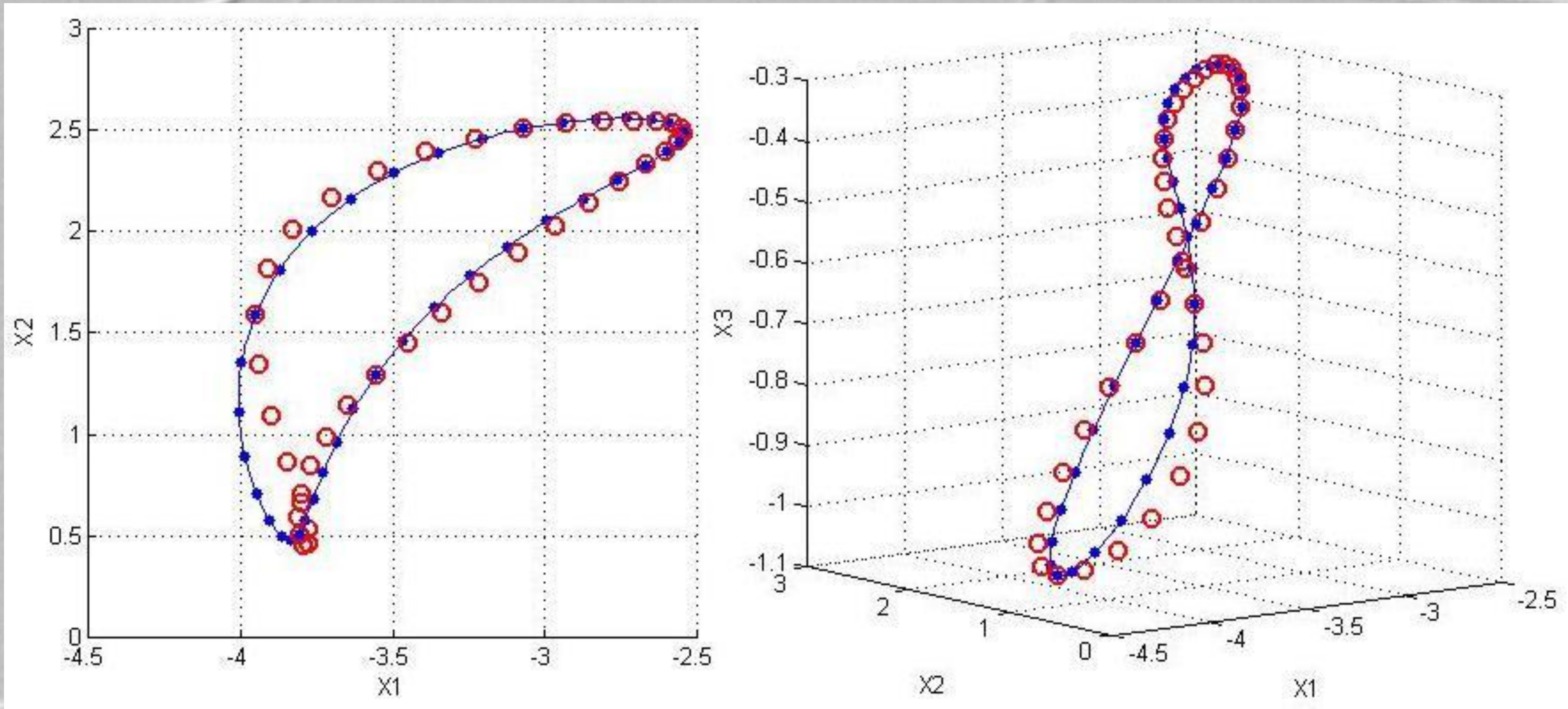
Results: Initial Guess 4



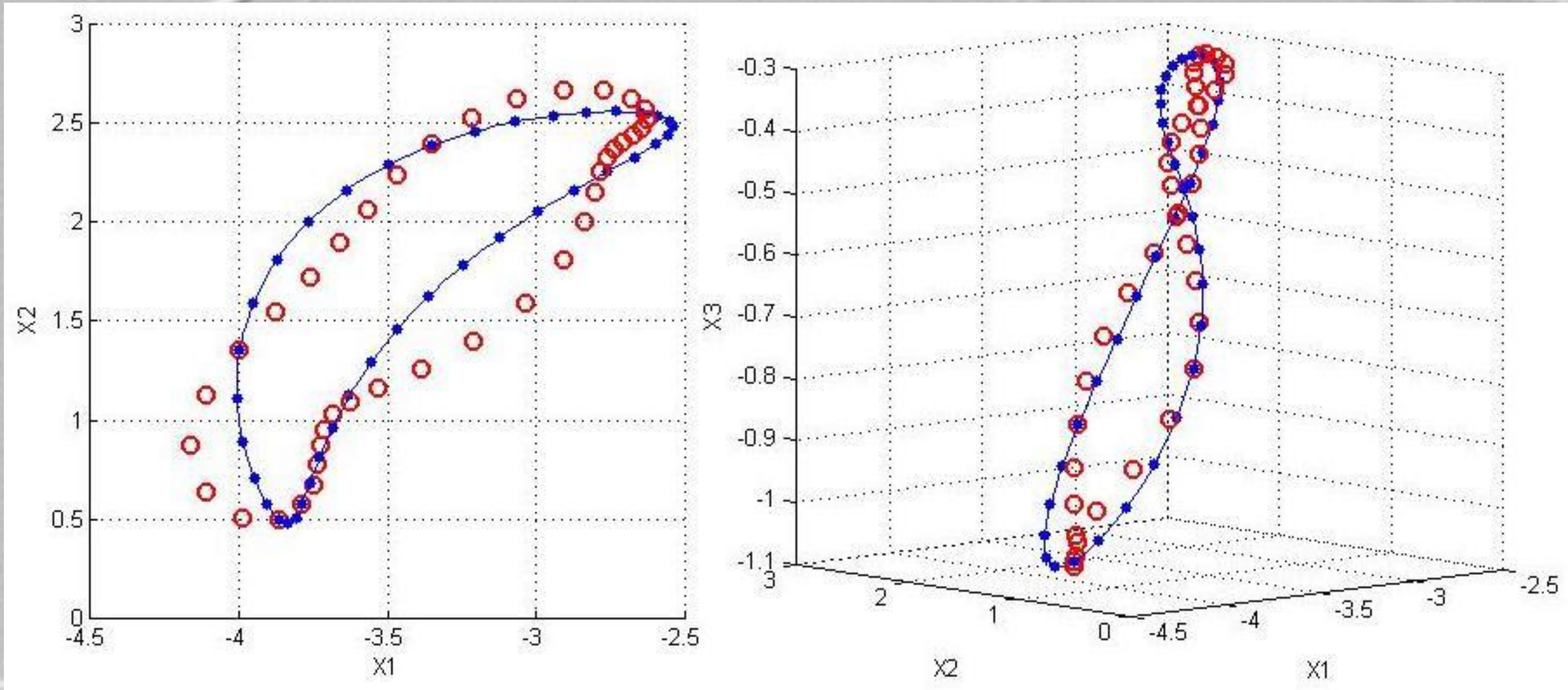
Results: Initial Guess 5



Results: Initial Guess 6



Results: Initial Guess 7



Conclusions

- A new approximate synthesis algorithm was developed minimizing the total deviation, *d*, from specified poses represented as points in the kinematic mapping image space.
- No heuristics are necessary and only five variables are needed.
- The algorithm returns a list of best generating mechanisms ranked according to *d*.
- The minimization could be further developed to jump from local minima to other local minima depending on desired "closeness" to specified poses.
- Relationships between the surface shape parameters may be exploited so the algorithm recognizes undesirable solutions and avoids iterations in those directions.

MECH 5507
Advanced Kinematics

Applications to Analysis

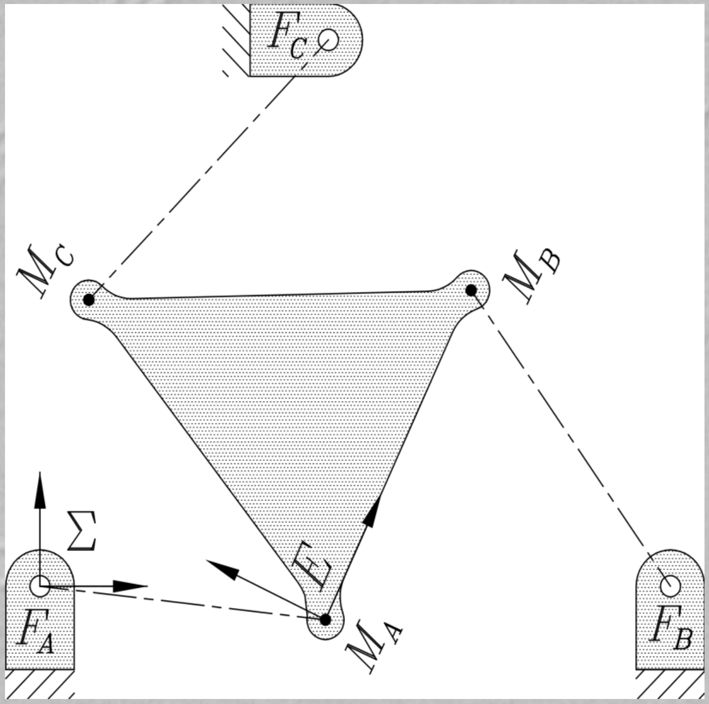
Professor M.J.D. Hayes
Department of Mechanical and
Aerospace Engineering

Applications to Analysis

- Kinematic mapping can also be used effectively for the analysis of complex kinematic chains.
- A very common example is a planar three-legged manipulator.
- A moving rigid planar platform connected to a fixed rigid base by three open kinematic chains. Each chain is connected by 3 independent 1 DOF joints, one of which is active.

General Planar Three-Legged Platforms

- 3 arbitrary points in a particular plane, described by frame *E*, that can have constrained motion relative to 3 arbitrary points in another parallel plane, described by frame Σ.
- Each platform point keeps a certain distance from the corresponding base point. These distances are set by the variable joint parameter and the topology of the kinematic chain.

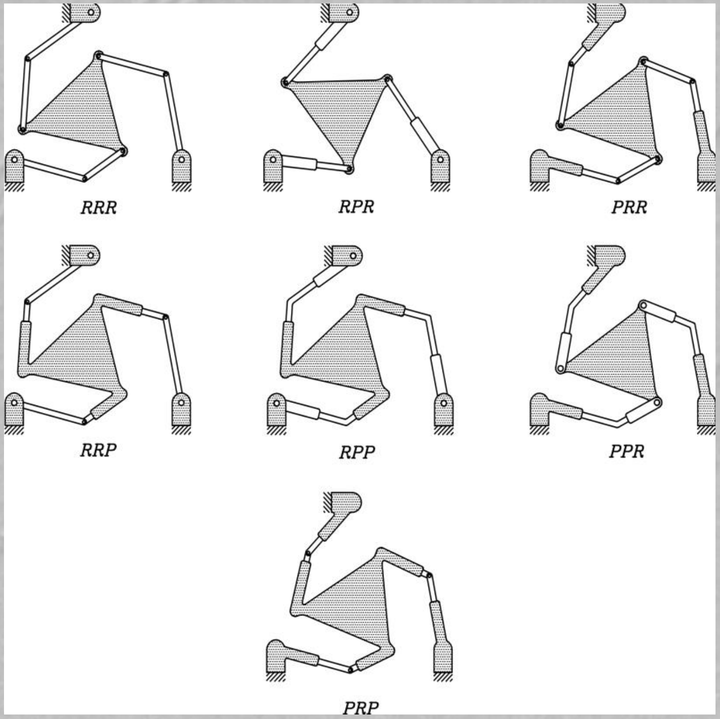


Characteristic Chains

- The possible combinations of *R* and *P* pairs of 3 joints starting from the fixed base are:

  RRR, *RPR*, *RRP*, *RPP*, *PRR*,
  PPR, *PRP*, *PPP*

- The *PPP* chain is excluded since no combination of translations can cause a rotation.
- 7 possible topologies each characterized by one simple chain.



Passive Sub-Chains

- There are 21 possible joint actuation schemes, as any of the 3 joints in any of the 7 characteristic chains may be active.
- When the active joint input is set, the remaining passive sub-chain is one of the following 3:

RR, *PR*, *RP*

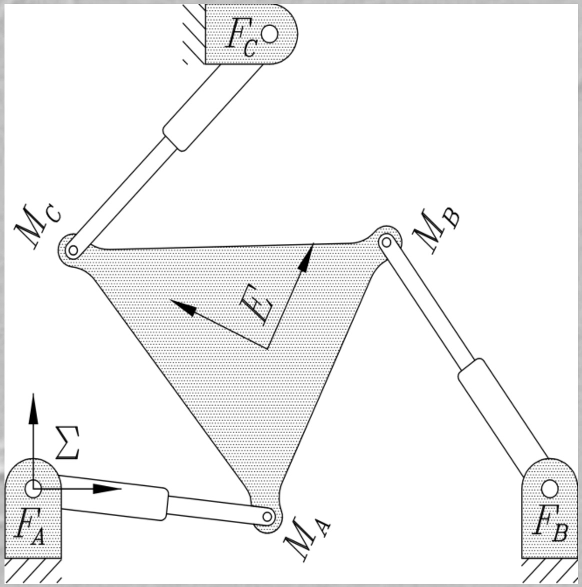
- The *PP*-type sub-chains are disregarded because platforms containing such sub-chains are more likely to be architecture singular.
- Thus, the number of different three-legged platforms is

$$C(n,k) = \frac{(n+k-1)!}{k!(n-1)!} \Rightarrow C(18,3) = 1140$$

- The direct kinematic analysis of all 1140 types is possible with this method.

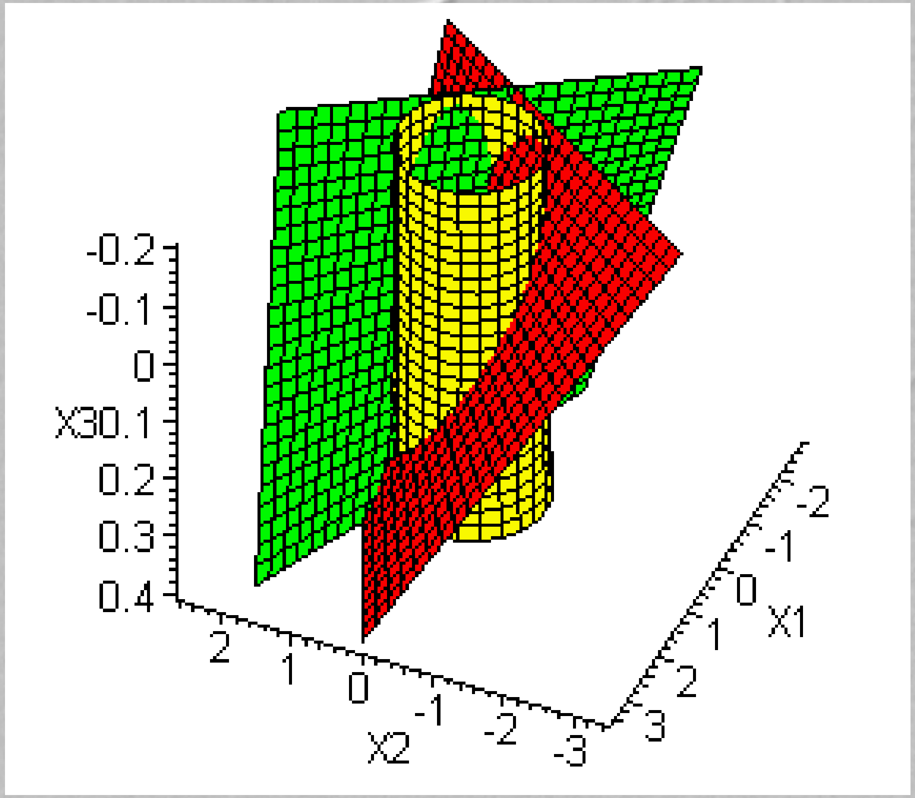
Kinematic Constraints

- *RR*-type legs: hyperboloid
  - One of the passive *R*-pairs has fixed position in Σ. The other, with fixed position in *E*, moves on a circle of fixed radius centred on the stationary *R*-pair.
- *PR*-type legs: hyperbolic paraboloid
  - The passive *R*-pair, with fixed position in *E*, is constrained to move on a line with fixed line coordinates in Σ.
- *RP*-type legs: hyperbolic paraboloid
  - The passive *P*-pair, with fixed position in *E*, is constrained to move on a point with fixed point coordinates in Σ. These are kinematic inversions, or projective duals, of the *PR*-type platforms.

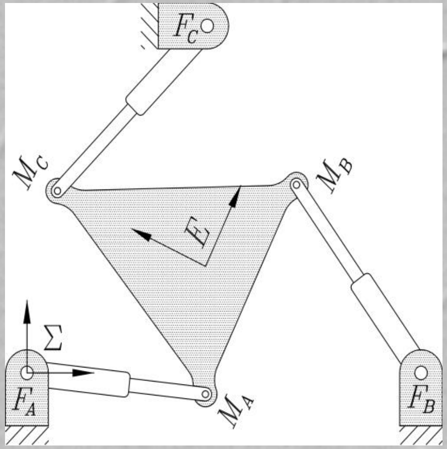


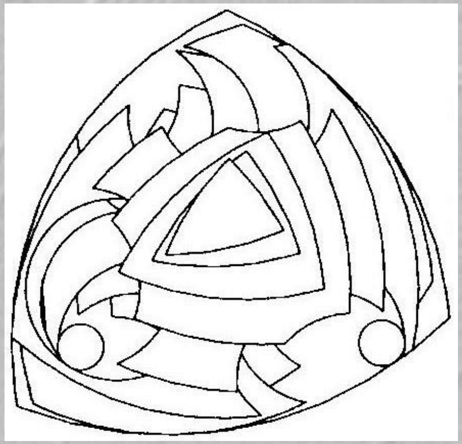
The Direct Kinematic Problem

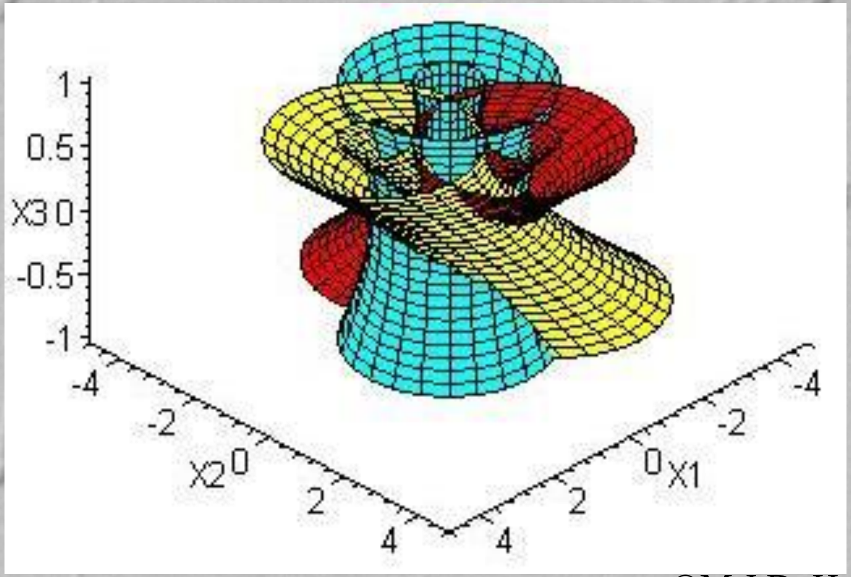
- The direct kinematic position analysis of any planar three-legged platform jointed with lower-pairs reduces to evaluating the points common to three quadric surfaces.

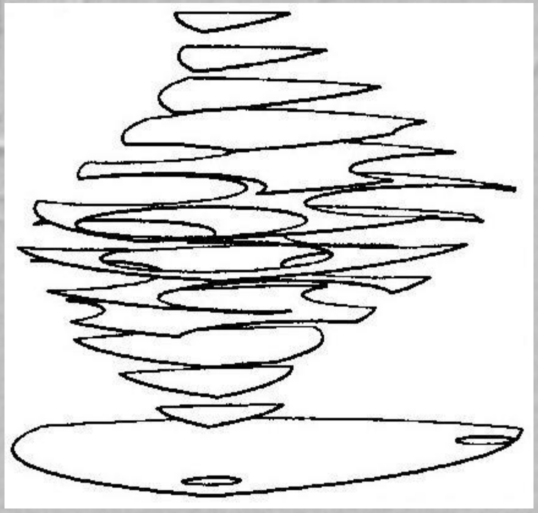


Workspace Visualization: Three *R<u>P</u>R*-Type Legs

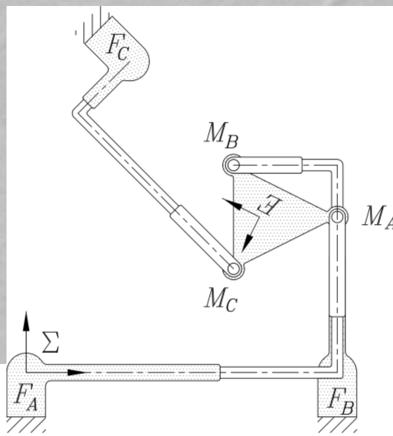


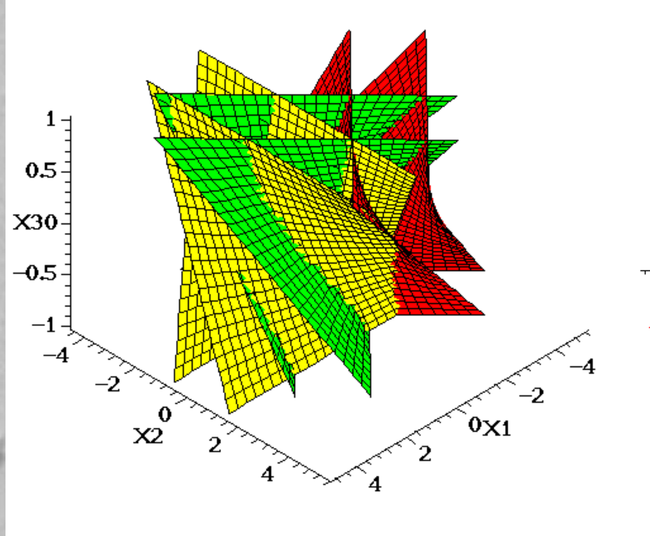


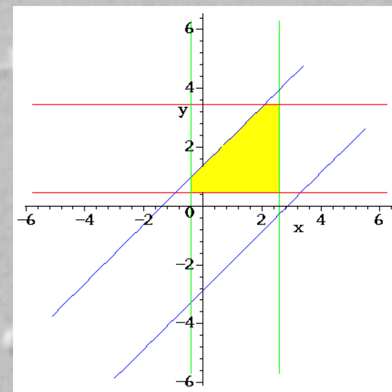




Three *P<u>P</u>R*-Type Legs







Cartesian layer $\phi = 150^o$

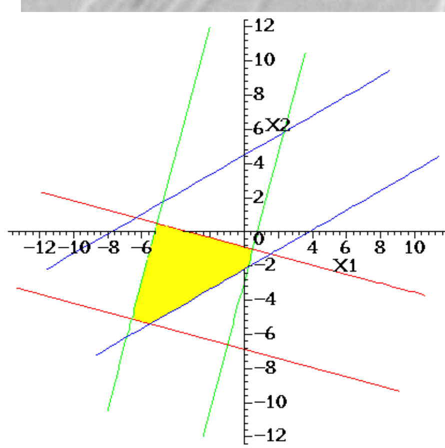
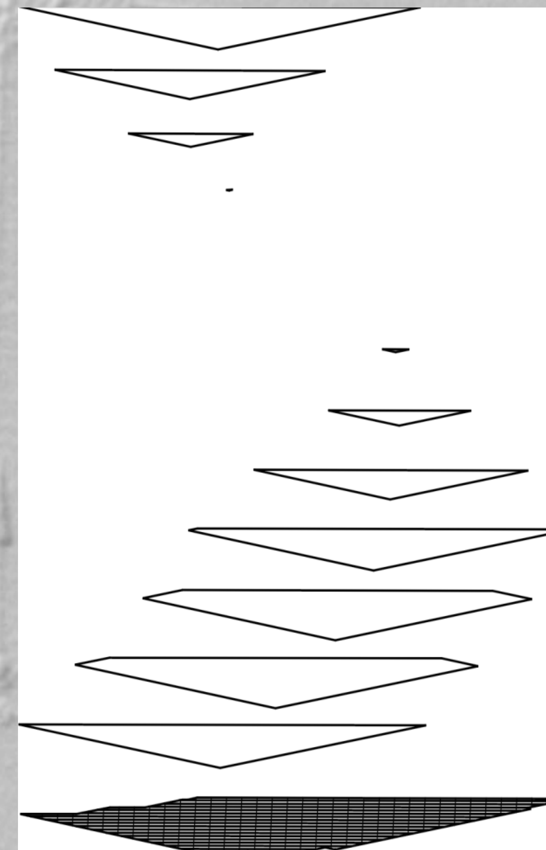
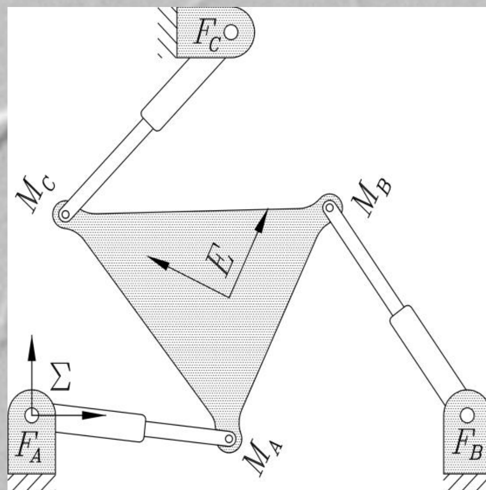
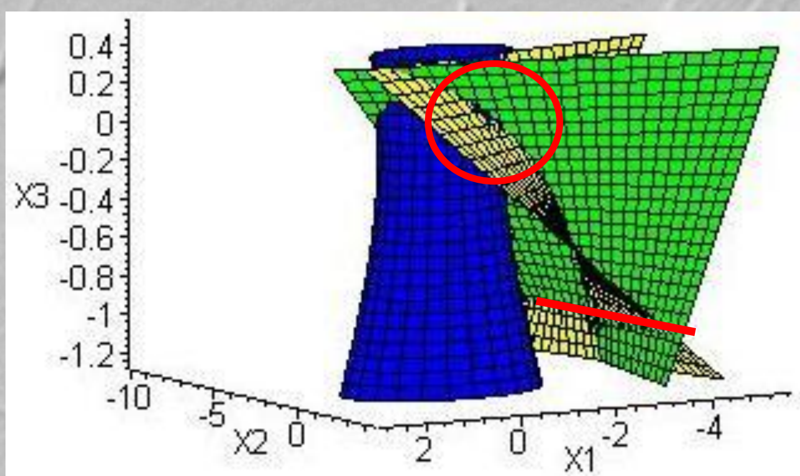


Image space layer $\phi = 150^o$



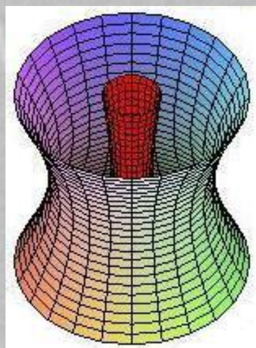
Mixed Leg (*RPR*, *RPR*, *RPR*) Platform

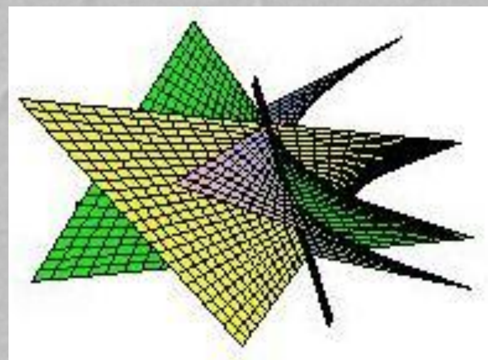


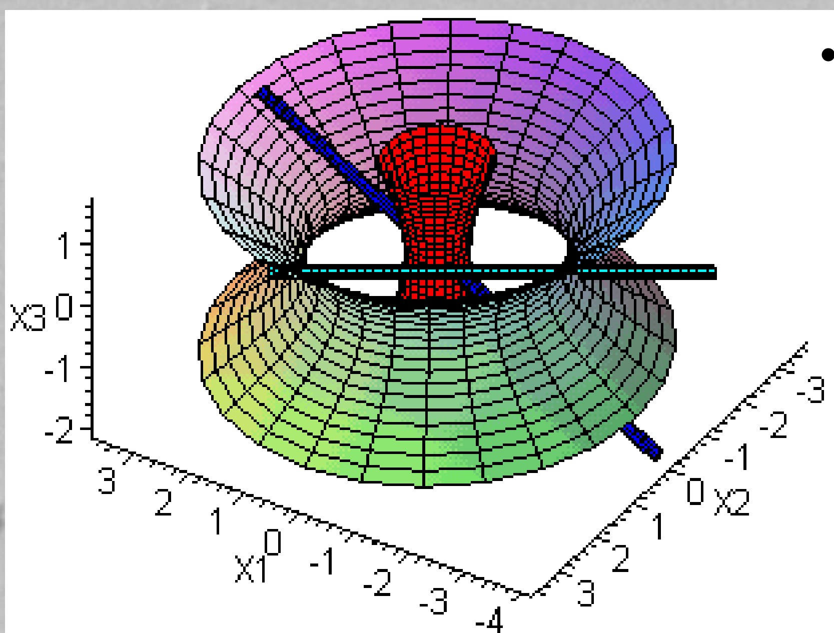


- This particular platform consists of one each of *RR*-type, *RP*-type and *PR*-type legs.
- The constraint surfaces for given leg inputs define the 3 constraint surfaces.
- The surfaces reveal 2 real and a pair of complex conjugate FK solutions.
- The *RPR* and *RPR* constraint surfaces have a common generator.

Mixed Leg Platform Workspace







- *RR*-type legs result in families of hyperboloids of one sheet all sharing the same axis.
- *PR*- and *RP*-type legs in general result in families of hyperbolic paraboloids.
- These families are pencils:
  - If the active joint is a *P*-pair the hyperbolic paraboloids in one family share a generator on the plane at infinity.
  - If the active joint is an *R*-pair the hyperbolic paraboloids in one family share a finite generator.