MECH 5507
ADVANCED KINEMATICS
Lecture Notes

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Chapter 1

Mechanisms

A mechanism is a device that transforms one motion into another. Many subclasses of mechanisms exist. For instance:

Machines: Mechanisms which transmit substantial forces, applying power, or changing its direction. Terms such as *force, torque, work* and *power* describe the predominant concepts. e.g Differential, clutch, brake.

Engines: When generated forces are associated with the conversion of energy of high temperature fluids to shaft power.

While a mechanism can be used to transmit force and power, the predominant associated concepts involve attaining a desired motion. Thus it may be defined as an assemblage of *rigid bodies* coupled by mechanical constraints such that there can be relative motion between them.

There is a direct analogy between the terms *structure, mechanism* and *machine* and to the branches of the science of mechanics.

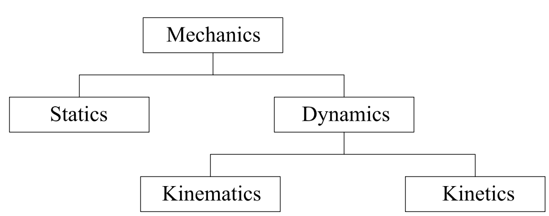

Figure 1.1: Branches of Mechanics.

Mechanics has two main constituents: statics and dynamics.

Statics: Analysis of systems of rigid bodies whose configuration is invariant with time.

Dynamics: Analysis of systems of rigid bodies whose relative configuration changes with time.

Dynamics, in turn, has two main components: kinematics and kinetics.

Kinematics: The study of motion, without considering the forces causing the motion: position, orientation, displacement, velocity, acceleration...

Kinetics: Equates motions to the forces that cause them.

Statics concerns structures: systems of rigid bodies that cannot move relative to each other. Kinematics concerns mechanisms. Kinetics concerns machines and engines.

Because kinematics is independent from the inertial properties of rigid bodies, it can be considered as purely geometric. Geometric configurations can always be represented, and manipulated, algebraically. The secret to understanding kinematics is to understand geometry, and to understand that if a particular geometric attribute is difficult to visualize, it can always be transformed, using algebra, to a different geometric representation that is easier to see.

It is also important to understand that algebra is simply rationalized geometry: that is, geometry without the pictures.

The *rigid body* assumption (i.e. the distance between every pair of points on the body is constant during a motion) is the key that allows us to consider kinematics and kinetics separately. For flexible bodies, the shape of the bodies themselves, and therefore their motions, are force dependent. Kinetics and kinematics must be considered simultaneously, significantly increasing the complexity of analysis and synthesis.

Even though there are no absolutely rigid bodies in reality, we can, when applied loads are known apriori, design the components to behave as rigid relative to the loads. Thus the rigid body assumption can usually always be justified.

1.1 Analysis and Synthesis

There are two fundamentally opposite aspects of the study of mechanisms: analysis and synthesis.

Synthesis: A process whose output prescribes the sizes, shapes, material (when considering *kinetic* synthesis . . . we will only look at *kinematic* synthesis), and arrangements of components so that the resulting mechanism will perform the desired task.

Analysis: Techniques allowing for quantifiable evaluation of an existing, or proposed mechanism.

As engineers, most of our effort is directed towards analysis, but the goal is always synthesis! Moreover, analysis is a tool that must be used to evaluate a synthesized mechanism.

Circularity, as we shall see, is vitally important to kinematics.

1.2 Kinematic Chains

A more general, yet more precise, term for *mechanism* is *kinematic chain*. A kinematic chain is defined as: a collection of rigid bodies coupled by mechanical constraints such that there can be relative motion between the rigid bodies.

The individual rigid bodies are called *links* in the chain. The kinematic chains are classified according to how the links are connected.

1.2.1 Simple Kinematic Chains

A kinematic chain is *simple* if each link in the chain is coupled to *at most* two other links. The degree of connectivity (DOC) of a link indicates the number of rigid bodies joined to it. If all links are binary ($DOC = 2$) the simple chain is *closed*. E.g. a four-bar mechanism. Alternately the simple chain is *open* with

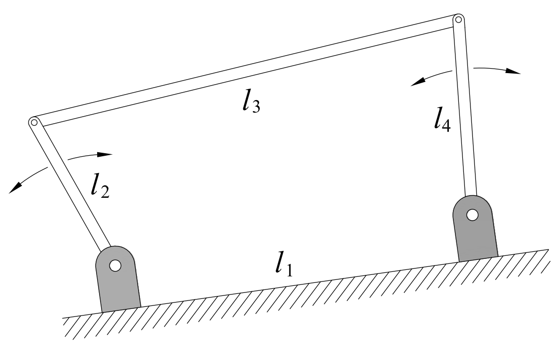

Figure 1.2: A typical planar 4-bar mechanism is a simple closed chain.

Figure 1.3: A typical *serially-connected* six-axis robot is a simple open chain.

the first and last links having $DOC = 1$. Figures 1.2 and 1.3 illustrate closed and open simple kinematic chains, respectively.

1.2.2 Complex Kinematic Chains

A chain is complex if *at least* one of its links has a $DOC \geq 3$. A complex kinematic chain may always be decomposed into simple kinematic sub-chains. Planar three-legged platforms and Stewart-Gough type six-legged spatial platforms (flight simulators) are typical examples of complex kinematic chains.

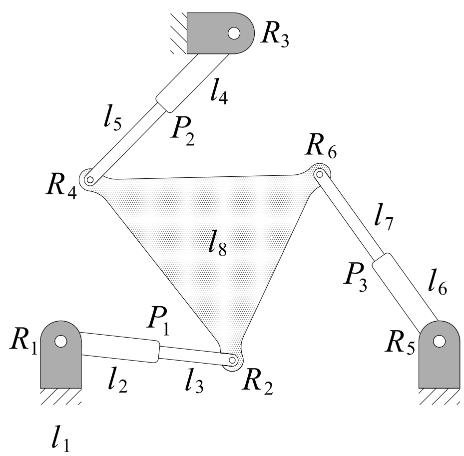

Figure 1.4: Complex kinematic chain. Central triangular-shaped link and fixed base link have $DOC = 3$.

A somewhat less typical example is the Peaucellier inversor. In the configuration shown in Figure 1.5, the linkage converts circular motion to straight line motion, without sliding.

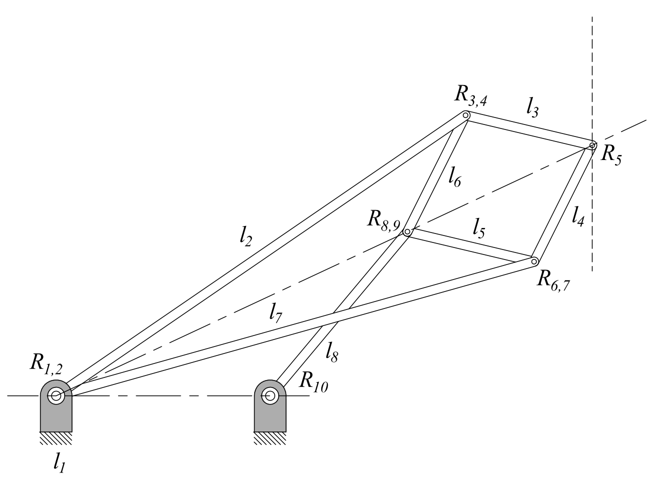

Figure 1.5: Peaucellier inversor. Eight-bar linkage from 1864.

When lengths $AD = DC$, the motion of point C on the circle centered at D is mapped to the motion of point P along a straight line. When link CD is removed, the motion of points C and P are algebraic inverses.

Other famous *approximate* straight line linkages were developed by James Watt, Richard Roberts, and Pafuntlij Chebyshev in the 1700's and 1800's [1].

1.3 Degree-of-Freedom

The degree-of-freedom (DOF) of a kinematic chain is defined to be an integer value corresponding to the minimum number of independent parameters (notice I don't say *coordinates*) required to fully describe, geometrically, an arbitrary configuration of the output link in the chain. There is one parameter usually, though not always, defined on the field of real numbers, associated with each DOF. Because any one of these parameters may be changed without necessitating a change in any of the others, they are all independent. Such parameters are historically called *generalized coordinates*. For the study of kinematics, generalized coordinates usually represent measures of distance and angle.

A kinematic chain constitutes a mechanism if its DOF is greater than zero. It is called a *constrained mechanism* if the chain is closed *and* its DOF is identically equal to 1 (e.g. a four-bar mechanism). A general mechanism can have more than one DOF, and need not be a closed chain (e.g. a serially connected six-axis robot). The kinematic chain is a statically determinate structure if the DOF is identically zero. The kinematic chain is a hyper-static structure if the DOF is less than zero. Such structures are termed statically under-determined, or over-constrained.

An unconstrained rigid body free to move in 3-dimensional space has six DOF. The DOF are generally taken to be three translations parallel to three linearly independent basis directions and three rotations about three linearly independent axes, although any system of six *generalized coordinates* is sufficient. That is, the six numbers need not be three distances and three angles, nor do the coordinate axes need to be orthogonal, see *Kinematic Geometry of Mechanisms* [2]. We see that, philosophically, the upper bound on DOF is 6, while there is no lower bound. While it is not necessary that the rotation axes be parallel to the translation directions, it is usually convenient to use a 3D orthogonal reference frame to describe the space of a motion, the rotation axes respectively parallel to the coordinate axes.

Mechanical constraints are imposed on rigid bodies to limit their motion as required. In this sense, constraints are the complements of DOF. For instance, if a rigid-body has two DOF in Euclidean space, indicated by E_3, four constraints must be imposed.

1.4 Kinematic Pairs

The term *kinematic pair* indicates a joint between *two* links, hence the use of the word *pair*. Joints are mechanical constraints imposed on the links. Those involving surface contact are called *lower pairs*. Those normally involving point, line, or curve contact are *higher pairs*. Lower pairs enjoy innate practical advantages over higher pairs.

1. Applied loads are spread continuously over the contacting surfaces.

2. They can, in general, be more easily and accurately manufactured.

1.4.1 Lower Pairs

There are six types of lower pair (see Figure 1.6) classified in the following way.

1. **R-Pair.** The revolute R-pair is made up of two congruent mating surfaces of revolution. It has one rotational DOF about its axis.

2. **P-Pair.** The prismatic P-pair comprises two congruent non-circular cylinders, or prisms. It has one translational DOF.

3. **H-Pair.** The helical H-pair, or screw, consists of two congruent helicoidal surfaces whose elements are a convex screw and a concave nut. For an angle θ of relative rotation about the screw axis there is a coupled translation of distance S in a direction parallel to the screw axis. The sense of translation depends on the *hand* of the screw threads and on the sense of rotation. The distance S is the thread *pitch* for a rotation of $\theta = 360°$. When $S = 0$ it becomes an R-pair; when $S = \infty$ it becomes a P-pair. The H-pair has one DOF specified as a translation or a rotation, coupled by the pitch S.

4. **C-Pair.** The cylindrical C-pair consists of mating convex and concave circular cylinders. They can rotate relative to each other about their common axis, and translate relative to each other in a direction parallel to the axis. Hence the C-pair has two DOF: one rotational, the other translational.

5. **S-Pair.** The spherical S-pair consists of a solid sphere which exactly conforms with a spherical shell. They are also called *ball-joints*. S-pairs permit three rotational DOF about intersecting orthogonal axes.

6. **E-Pair.** The planar E-pair (for the German word "ebene", meaning "plane") is a special S-pair comprising two concentric spheres of infinite radius. They permit two orthogonal translations and one rotational DOF about an axis orthogonal to the plane of translation. They provide three DOF in total.

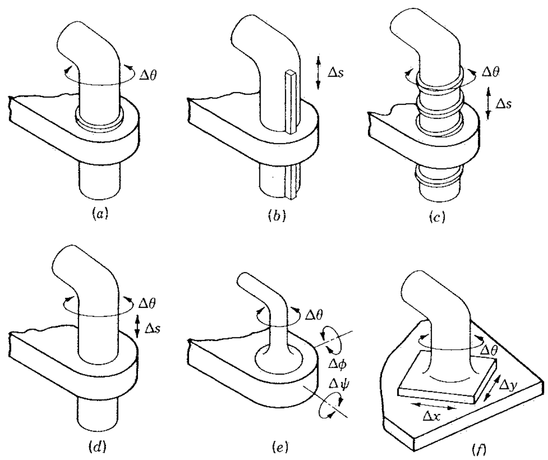

Figure 1.6: The six lower pairs: (a) revolute or pin; (b) prismatic; (c) helical; (d) cylindrical; (e) spherical; (f) planar.

Table 1.1: Summary of the lower pairs and their respective DOF.

Pair	DOF
R	1
P	1
H	1
C	2
S	3
E	3

1.4.2 Higher Pairs

Higher pairs are important because they often offer the most direct means of achieving complicated motions. The main drawback is that they are typically more complicated, implying that they are more expensive to design and manufacture. A few examples are mating spur gears, rack and pinion, cam and follower. The higher pairs may be classified according to the nature of the relative motion between the jointed links.

1. **Pure Sliding.** The relative motion is pure translation as in a reciprocating cam activating a knife-edge or mushroom head follower, or the finger tip of a robot hand sliding on a flat surface. See Figures 1.7 (a) and (b), as well as Figure 1.8 (a), for example.

2. **Pure Rolling.** The relative motion involves rolling without slip, such as the tangential pitch circles or mating sets of spur gears, or rack and pinion system, see Figure 1.7 (c) and Figure 1.8 (b).

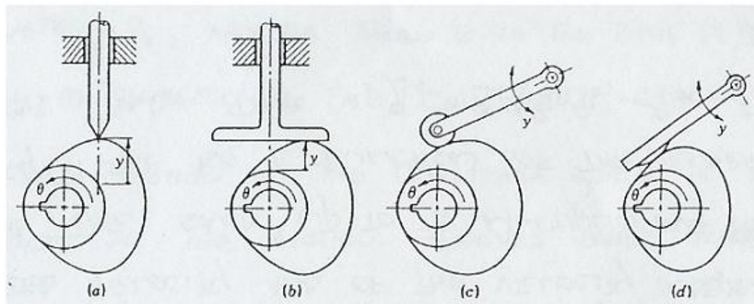

Figure 1.7: Plate cams with (a) an offset reciprocating knife-edge follower; (b) a reciprocating flat-face follower; (c) an oscillating roller follower; and (d) an oscillating curved-shoe follower.

3. **Combination of Sliding and Rolling.** In rotating cam and follower systems the tip of the follower slides along any constant radius of curvature portions of the cam surface. As the cam rotates and, relative to the follower, its radius of curvature changes, the follower rotates about the same axis. As this occurs, the follower tip will also roll on the cam surface. This can be observed in Figure 1.7 (d).

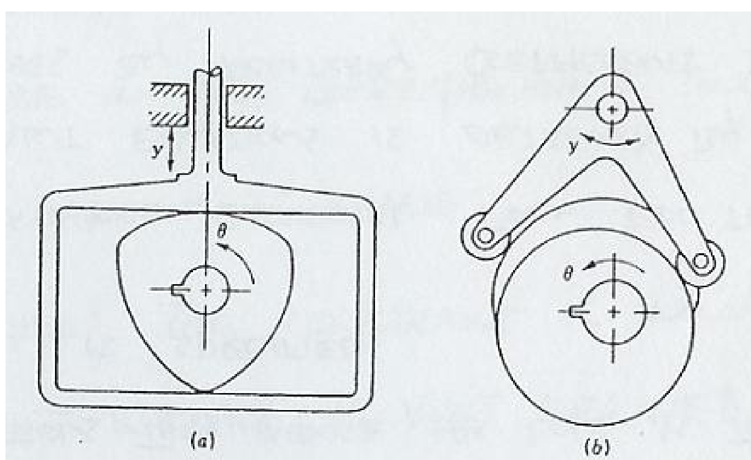

Figure 1.8: (a) A constant-breadth cam with a reciprocating flat-face follower. (b) Conjugate cams with an oscillating roller follower.

1.5 Holonomic and Non-Holonomic Constraints

The term *holonomic* is derived from the Greek word *holos* meaning *integer*. It describes constraints that may be expressed in *integral form*, i.e., in terms of displacements, as opposed to *differential form*, i.e., in terms of linear and angular velocities. Differential form kinematic constraints involving link angular velocities are generally non-holonomic unless the motion is planar and occurs without slip, see any text on advanced dynamics, for example *Advanced Engineering Dynamics* by J.H. Ginsberg [3].

The i^{th} of n constraint equations confining the motion of a rigid body can be written in the form of a function:

$$f_i(q_1, q_2, ..., q_m, t) \quad = \quad 0, \tag{1.1}$$

where the m q's are constrained generalized coordinates, t stands for time, and the subscript i indicates a particular constraint equation. Any limitation placed on the generalized coordinates restricts the position of the rigid body, and hence are called *position constraints*.

Position constraints impose restrictions on the velocity as well. The velocity constraints are obtained by differentiating Equation (1.1) with respect to time:

$$\dot{f}_i = \frac{df_i}{dt} = \sum_{j=1}^{m} \left[\frac{\partial}{\partial q_j} f_i(q_1, q_2, ..., q_m, t) \right] \dot{q}_j + \left[\frac{\partial}{\partial t} f_i(q_1, q_2, ..., q_m, t) \right] = 0, \tag{1.2}$$

where the $\dot{q}$'s are called *generalized velocities* (i.e. dq_i/dt). Equations (1.1) and (1.2) are equivalent in the limitations they impose, as long as the initial position and orientation are specified.

A more general form for the velocity constraint equations is obtained by replacing the derivatives by arbitrary coefficients that are functions of only the generalized coordinates and time:

$$\sum_{j=1}^{m} a_{ij}(q_1, q_2, ..., q_m, t)\dot{q}_j + b_i(q_1, q_2, ..., q_m, t) \quad = \quad 0. \tag{1.3}$$

Equations (1.2) and (1.3) represent equivalent constraints if the corresponding coefficients of each generalized velocity and of the velocity independent term are the same up to a multiplicative factor, which may itself be a function of the generalized coordinates and time, $g_i = g_i(q_1, q_2, ..., q_m, t)$. A velocity constraint is derivable from a position constraint, and vice-versa, if and only if:

$$a_{ij} g_i \quad = \quad \frac{\partial f_i}{\partial q_j}, \;\; b_i g_i \quad = \quad \frac{\partial f_i}{\partial t}. \tag{1.4}$$

The velocity constraint equations are holonomic (meaning integrable) if they satisfy Equation (1.4), otherwise they are non-holonomic.

This terminology actually refers to the differential a constraint equation called the *Pfaffian form* [4]. The Pfaffian form is obtained by multiplying Equation (1.3) through by dt, giving:

$$\sum_{j=1}^{m} a_{ij}(q_1, q_2, ..., q_m, t)dq_j + b_i(q_1, q_2, ..., q_m, t)dt \quad = \quad 0. \qquad (1.5)$$

When Equation (1.4) is satisfied, multiplying Equations (1.5) by the g_i functions transform the Pfaffian forms to perfect differentials of each function f_i. This leads to the following definition:

> **Definition:** A velocity constraint is holonomic if there exists an integrating factor g_i for which the Pfaffian form of the constraint equation becomes a perfect differential. In this case, it may be integrated yielding the position constraint on the generalized coordinates.

The concept of a holonomic constraint may be viewed from a geometric perspective. The generalised coordinates q_i may be taken to be the basis $(q_1, q_2, ..., q_m)$ of an m-dimensional (mD) constraint space. The constrained motion in the constraint space is a locus of points as the motion evolves over time. Consider a holonomic constraint $f_i(q_1, q_2, ..., q_m, t) = 0$. At any instant in time t the position of the rigid body is confined to some surface in the constraint space. The corresponding Pfaffian form of the velocity constraints states that infinitesimal displacements *must* be in the corresponding tangent plane to the constraint surface at that point.

When the constraint is non-holonomic, the constraint surface is *not* defined. Hence, the velocity constraint cannot be integrated. In this case the Pfaffian form of the constraint equation restricts infinitesimal displacements to lie in a plane that can only be defined by the current state of motion and the position level kinematics cannot be directly obtained from the velocities by integration.

Bibliography

[1] J.E. Shigley and J.J. Uicker, Jr. *Mechanism and Machine Theory, 2nd ed.* McGraw-Hill, New York, N.Y., U.S.A., 1995.

[2] K.H. Hunt. *Kinematic Geometry of Mechanisms.* Clarendon Press, Oxford, England, 1978.

[3] J.H. Ginsberg. *Advanced Engineering Dynamics.* Harper & Row, Publishers, New York, N.Y., U.S.A., 1988.

[4] V. V. Sychev. *The Differential Equations Of Thermodynamics.* CRC Press, Boca Raton, Fla., U.S.A., 1991.