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Replacing Pfaffians and applications

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Abstract

We present some new Pfaffian identities related to the Plücker relations. As consequences we obtain a quadratic identity for the number of perfect matchings of plane graphs, which has a simpler form than the formula by Yan et al. [W.G. Yan, Y.-N. Yeh, F.J. Zhang, Graphical condensation of plane graphs: A combinatorial approach, Theoret. Comput. Sci. 349 (2005) 452–461], and we also obtain some new determinant identities.

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1. Introduction

Let $A = (a_{ij})_{n\times n}$ be a skew symmetric matrix of order n, where n is even. Suppose that $\pi = \{(s_1, t_1), (s_2, t_2), \dots, (s_{\frac{n}{2}}, t_{\frac{n}{2}})\}$ is a partition of $[n]$, that is, $[n] = \{s_1, t_1\} \cup \{s_2, t_2\} \cup \cdots \cup \{s_{\frac{n}{2}}, t_{\frac{n}{2}}\}$, where $[n] = \{1, 2, \dots, n\}$. Define

$$b_\pi = \operatorname{sgn}(s_1 t_1 s_2 t_2 \dots s_{\frac{n}{2}} t_{\frac{n}{2}}) \prod_{l=1}^{n/2} a_{s_l t_l},$$

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where $\operatorname{sgn}(s_1t_1s_2t_2\ldots s_{\frac{n}{2}}t_{\frac{n}{2}})$ denotes the sign of the permutation $s_1t_1s_2t_2\ldots s_{\frac{n}{2}}t_{\frac{n}{2}}$. Note that b_π depends neither on the order in which the classes of the partition are listed nor on the order of the two elements of a class. So b_π indeed depends only on the choice of the partition π. The Pfaffian of A, denoted by $Pf(A)$, is defined as

$$Pf(A)=\sum_{\pi} b_\pi,$$

where the summation is over all partitions of $[n]$ which are of the form of π. For the sake of convenience, we define the Pfaffian of A to be zero if A is a skew symmetric matrix of odd order. The following result is well known (see [1]):

Cayley's Theorem. *For any skew symmetric matrix $A=(a_{ij})_{n\times n}$ of order n, we have*

$$\det(A)=\left[Pf(A)\right]^2.$$

Suppose that $G=(V(G),E(G))$ is a weighted graph with the vertex set $V(G)=\{1,2,\ldots,n\}$, the edge set $E(G)=\{e_1,e_2,\ldots,e_m\}$, and the edge-weight function $\omega\colon E(G)\to\mathcal{R}$, where $\omega(e):=\omega_e=a_{ij}$ $(\neq 0)$ if $e=(i,j)$ is an edge of G and $\omega_e=a_{ij}=0$ otherwise, and $\mathcal{R}$ is the set of real numbers. Suppose G^e is an orientation of G. Let $A(G^e)=(b_{ij})_{n\times n}$ be the matrix of order n defined as follows:

$$b_{ij}=\begin{cases} a_{ij} & \text{if } (i,j) \text{ is an arc in } G^e,\\ -a_{ij} & \text{if } (j,i) \text{ is an arc in } G^e,\\ 0 & \text{otherwise.}\end{cases}$$

The matrix $A(G^e)$ is called the skew adjacency matrix of G^e (see [18]). Obviously, $A(G^e)$ is a skew symmetric matrix, that is, $(A(G^e))^T=-A(G^e)$.

Given a skew symmetric matrix $A=(a_{ij})_{n\times n}$ with n even, let $G=(V(G),E(G))$ be a weighted graph with the vertex set $V(G)=\{1,2,\ldots,n\}$, where $e=(i,j)$ is an edge of G if and only if $a_{ij}\neq 0$, and the edge-weight function is defined as $\omega_e=|a_{ij}|$ if $e=(i,j)$ is an edge of G and $\omega_e=0$ otherwise. Define G^e as the orientation of G in which the direction of every edge $e=(i,j)$ of G is from vertices i to j if $a_{ij}>0$ and from vertices j to i otherwise. We call G^e to be the corresponding directed graph of A. Obviously, $A=A(G^e)$. A perfect matching of a graph G is a set of independent edges of G covering all vertices of G. It is not difficult to see that the Pfaffian $Pf(A)$ of A can be defined as

$$Pf(A)=\sum_{\pi\in\mathcal{M}(G)} b_\pi,$$

where the summation is over all perfect matchings $\pi=\{(s_1,t_1),(s_2,t_2),\ldots,(s_{\frac{n}{2}},t_{\frac{n}{2}})\}$ of G, and b_π is the product of all $\omega_{(s_i,t_i)}$ for $1\leqslant i\leqslant\frac{n}{2}$.

Pfaffians have been studied for almost two hundred years (see [13,29] for a history), and continue to find numerous applications, for example in matching theory [18] and in the enumeration of plane partitions [29]. It is interesting to extend Leclerc's combinatorics of relations for determinants [16] to the analogous rules for Pfaffians. Using tools from multilinear algebra, Dress and Wenzel [4] gave an elegant proof of an identity concerning Pfaffians of skew symmetric matrices, which yields the Grassmann–Plücker identities (for more details see [33, Section 7]). Okada [23] presented a Pfaffian identity involving elliptic functions, whose rational limit gives a generalization of Schur's Pfaffian identity. Knuth [13] used a combinatorial method to give an elegant proof of a classical Pfaffian identity found in [30]. Hamel [6] followed Knuth's approach and introduced other combinatorial methods to prove a host of Pfaffian identities from physics in [7,22,31]. Hamel also provided a combinatorial proof of a result in [28] and a new vector-based Pfaffian identity, and gave an application to the theory of symmetric functions by proving an identity for Schur Q-functions. For some related recent results, see also [8,9,19,24].

Problems involving enumeration of perfect matchings of a graph were first examined by chemists and physicists in the 1930s (for history see [18,25]), for two different (and unrelated) purposes: the study of aromatic hydrocarbons and the attempt to create a theory of the liquid state. Shortly after the advent of quantum chemistry, chemists turned their attention to molecules like benzene composed of carbon rings with attached hydrogen atoms. For these researchers, perfect matchings of a polyhex graph corresponded to Kekulé structures, i.e., assigning single and double bonds in the associated hydrocarbon (with carbon atoms at the vertices and tacit hydrogen atoms attached to carbon atoms with only two neighboring carbon atoms). There are strong connections between combinatorial and chemical properties for such molecules; for instance, those edges which are present in comparatively few of the perfect matchings of a graph turn out to correspond to the bonds that are least stable, and the more perfect matchings a polyhex graph possesses the more stable is the corresponding benzenoid molecule. The number of perfect matchings is an important topological index which had been applied for estimation of the resonant energy and total π-electron energy and calculation of the Pauling bond order. So far, many researchers have given most of their attention to counting perfect matchings of graphs. See for example papers [2,14,20,21,25–27,34,36].

This paper is inspired by two results, one of which is that we can use the Pfaffian method to enumerate perfect matchings of plane graphs (see [11,12]). That is, we can express the number of perfect matchings of a plane graph G in terms of the Pfaffian of the skew adjacency matrix of a Pfaffian orientation of G. Inspired by Dodgson's determinant-evaluation rule in [3] and the Plücker relations for Pfaffians, Propp [26], Kuo [14,15], and Yan and Zhang [36] obtained a method of graphical vertex condensation for enumerating perfect matchings of plane bipartite graphs. The second is that by using the Matching Factorization Theorem in [2], Yan et al. [34] found a method of graphical edge condensation for counting perfect matchings of plane graphs. It is natural to ask whether there exist some Pfaffian identities similar to the Plücker relations, which can result in some formulas for the method of graphical edge condensation for enumerating perfect matchings of plane graphs. The results (Theorem 3.1 and Corollary 3.2) in Section 3 answer this question in the affirmative. Particularly, we obtain two new determinant identities (Theorems 4.2 and 4.3) in Section 4.1, and as an important application in graph theory we prove a quadratic relation on graphical edge condensation for enumerating perfect matchings of plane graphs in Section 4.2 (Theorem 4.4), which has a simpler form than the formula in [34].

2. Some lemmas

In order to present the following lemmas, we introduce some notation and terminology. If I is a subset of $[n]$, we use A_I to denote the minor of A obtained by deleting rows and columns indexed by I. If $I = \{i_1, i_2, \ldots, i_l\} \subseteq [n]$ and $i_1 < i_2 < \cdots < i_l$, we use $Pf_A(i_1 i_2 \ldots i_l) =: Pf_A(I)$ to denote the Pfaffian of $A_{[n]\setminus I}$. Following Knuth's notation in [13], for two words α and β we define $s(\alpha, \beta)$ to be zero if either α or β has a repeated letter, or if β contains a letter not in α.

Otherwise, $s(\alpha, \beta)$ denotes the sign of the permutation that takes α into the word $\beta(\alpha \setminus \beta)$ (where $\alpha \setminus \beta$ denotes the word that remains when the elements of β are removed from α). Let S be a subset of $\{1, 2, \ldots, n\}$. For convenience, if $|S|$ is even (respectively odd), then we call S an even (respectively odd) subset of $[n]$.

Dress and Nenzel [4] used tools from multilinear algebra to prove a Pfaffian identity, which was found by Wenzel [33], as follows:

Lemma 2.1. *(See Wenzel [33] and Dress and Nenzel [4].) For any two subsets $I_1, I_2 \subseteq [n]$ of odd cardinality and elements $i_1, i_2, \ldots, i_t \in [n]$ with $i_1 < i_2 < \cdots < i_t$ and $\{i_1, i_2, \ldots, i_t\} = I_1 \triangle I_2 =: (I_1 \setminus I_2) \cup (I_2 \setminus I_1)$, if $A = (a_{ij})_{n\times n}$ is a skew symmetric matrix with n even, then*

$$\sum_{\tau=1}^{t} (-1)^{\tau} Pf_A\big(I_1 \triangle \{i_\tau\}\big) Pf_A\big(I_2 \triangle \{i_\tau\}\big) = 0.$$

A direct result of Lemma 2.1 is the following lemma, which will play an important role in the proofs of our main results.

Lemma 2.2. *Suppose that $A = (a_{ij})_{n\times n}$ is a skew symmetric matrix with n even, and α is an even subset of $[n]$. Let $\beta = \{i_1, i_2, \ldots, i_{2p}\} \subseteq [n] \setminus \alpha$, where $i_1 < i_2 < \cdots < i_{2p}$. Then, for any fixed $s \in [2p]$, we have*

$$Pf_A(\alpha) Pf_A(\alpha\beta) = \sum_{l=1}^{2p} (-1)^{l+s+1} Pf_A(\alpha i_s i_l) Pf_A(\alpha\beta \setminus i_s i_l),$$

where $Pf_A(\alpha i_s i_s) = 0$.

The following result is a special case of Lemma 2.2.

Corollary 2.1. *(See Tutte [32].) Suppose that $A = (a_{ij})_{n\times n}$ is a skew symmetric matrix and $\{i, j, k, l\} \subseteq [n]$. Then*

$$\begin{aligned} &Pf(A_{\{i,j,k,l\}}) Pf(A) \\ &\quad = Pf(A_{\{i,j\}}) Pf(A_{\{k,l\}}) - Pf(A_{\{i,k\}}) Pf(A_{\{j,l\}}) + Pf(A_{\{i,l\}}) Pf(A_{\{j,k\}}). \end{aligned} \tag{2.1}$$

There exists a formula for the determinant, called Dodgson's determinant-evaluation rule, similar to Corollary 2.1, as follows (see [3]):

$$\det(A_{\{1,n\}}) \det(A) = \det(A_{11}) \det(A_{nn}) - \det(A_{1n}) \det(A_{n1}), \tag{2.2}$$

where A is an arbitrary matrix of order n and A_{ij} is the minor of A obtained by deleting the ith row and the jth column.

The following result shows the relation between the Pfaffian and the determinant.

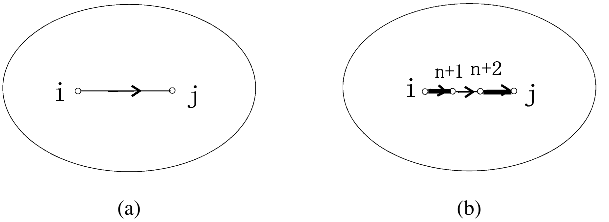

Fig. 1. (a) The directed graph G^e. (b) The directed graph $\overline{G}^e$.

Lemma 2.3. *(See Godsil [5].) Let A be a square matrix of order n. Then*

$$Pf\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} = (-1)^{\frac{1}{2}n(n-1)} \det(A).$$

Let $A = (a_{st})_{n\times n}$ be a skew symmetric matrix of order n and G^e the corresponding directed graph. Suppose (i, j) is an arc in G^e and hence $a_{ij} > 0$. Let $\overline{G}^e$ be a directed graph with vertex set $\{1, 2, \ldots, n+1, n+2\}$ obtained from G^e by deleting the arc (i, j) and adding three arcs $(i, n+1)$, $(n+1, n+2)$, and $(n+2, j)$ with weights $\sqrt{a_{ij}}$, 1, and $\sqrt{a_{ij}}$, respectively (see Fig. 1(a) and (b) for an illustration). For convenience, if $a_{ij} = 0$ we also regard $\overline{G}^e$ as a directed graph obtained from G^e by adding three arcs $(i, n+1)$, $(n+1, n+2)$, and $(n+2, j)$ with weights 0, 1, and 0. The following lemma will play a key role in the proofs of our main results.

Lemma 2.4. *Suppose that $A = (a_{st})_{n\times n}$ is a skew symmetric matrix and G^e is the corresponding directed graph. Let $\overline{G}^e$ be the directed graph with $n + 2$ vertices defined above and $A(\overline{G}^e)$ the skew adjacency matrix of $\overline{G}^e$. Then*

$$Pf(A) = Pf\big(A\big(\overline{G}^e\big)\big).$$

Proof. Let G and $\overline{G}$ be the underlying graphs of G^e and $\overline{G}^e$, and let $A(G^e)$ be the skew adjacency matrix of G^e. Hence $A(G^e) = (a_{st})_{n\times n}$ and $A(\overline{G}^e) = (b_{st})_{(n+2)\times(n+2)}$, where

$$b_{st} = \begin{cases} a_{st} & \text{if } 1 \leqslant s, t \leqslant n \text{ and } (s,t) \neq (i,j), (j,i), \\ \sqrt{a_{ij}} & \text{if } (s,t) = (i, n+1) \text{ or } (n+2, j), \\ -\sqrt{a_{ij}} & \text{if } (s,t) = (n+1, i) \text{ or } (j, n+2), \\ 1 & \text{if } (s,t) = (n+1, n+2), \\ -1 & \text{if } (s,t) = (n+2, n+1), \\ 0 & \text{otherwise.} \end{cases}$$

By the definitions above, we have

$$Pf(A) = Pf\big(A\big(G^e\big)\big).$$

Hence we only need to prove

$$Pf\big(A\big(G^e\big)\big) = Pf\big(A\big(\overline{G}^e\big)\big).$$

Note that, by the definition of the Pfaffian, we have

$$Pf\big(A\big(G^e\big)\big)=\sum_{\pi\in\mathcal{M}(G)} b_\pi,\qquad Pf\big(A\big(\overline{G}^e\big)\big)=\sum_{\overline{\pi}\in\mathcal{M}(\overline{G})} b_{\overline{\pi}},$$

where $\mathcal{M}(G)$ and $\mathcal{M}(\overline{G})$ denote the sets of perfect matchings of G and $\overline{G}$.

We partition the sets of perfect matchings of G and $\overline{G}$ as follows:

$$\mathcal{M}(G)=\mathcal{M}_1\cup\mathcal{M}_2,\qquad \mathcal{M}(\overline{G})=\overline{\mathcal{M}_1}\cup\overline{\mathcal{M}_2},$$

where $\mathcal{M}_1$ is the set of perfect matchings of G each of which contains edge $e=(i,j)$, $\mathcal{M}_2$ is the set of perfect matchings of G each of which does not contain edge $e=(i,j)$, $\overline{\mathcal{M}_1}$ is the set of perfect matchings of $\overline{G}$ each of which contains both of edges $(i,n+1)$ and $(n+2,j)$, and $\overline{\mathcal{M}_2}$ is the set of perfect matchings of $\overline{G}$ each of which contains edge $(n+1,n+2)$.

Suppose π is a perfect matching of G. If $\pi\in\mathcal{M}_1$, then there exists uniquely a perfect matching π' of $G-i-j$ such that $\pi=\pi'\cup\{(i,j)\}$. It is clear that there is a natural way to regard π' as a matching of $\overline{G}$. Define: $\overline{\pi}=\pi'\cup\{(i,n+1),(n+2,j)\}$. Hence $\overline{\pi}\in\overline{\mathcal{M}_1}$. Similarly, if $\pi\in\mathcal{M}_2$, we can define: $\overline{\pi}=\pi\cup\{(n+1,n+2)\}$ and hence $\overline{\pi}\in\overline{\mathcal{M}_2}$. It is not difficult to see that the mapping $f:\pi\mapsto\overline{\pi}$ between $\mathcal{M}(G)$ and $\mathcal{M}(\overline{G})$ is bijective.

Hence we only need to prove that for any perfect matching π of G we have $b_\pi=b_{\overline{\pi}}$. By the definition of $\overline{\pi}$, if $\pi=\{(s_1,t_1),(s_2,t_2),\dots,(s_{l-1},t_{l-1}),(i,j),(s_{l+1},t_{l+1}),\dots,(s_{\frac{n}{2}},t_{\frac{n}{2}})\}\in\mathcal{M}_1$, then $\overline{\pi}=\{(s_1,t_1),(s_2,t_2),\dots,(s_{l-1},t_{l-1}),(i,n+1),(n+2,j),(s_{l+1},t_{l+1}),\dots,(s_{\frac{n}{2}},t_{\frac{n}{2}})\}\in\overline{\mathcal{M}_1}$. Note that

$$\begin{aligned}&\operatorname{sgn}(s_1t_1\dots s_{l-1}t_{l-1}ijs_{l+1}t_{l+1}\dots s_{\frac{n}{2}}t_{\frac{n}{2}})\\&\quad=\operatorname{sgn}\big(s_1t_1\dots s_{l-1}t_{l-1}i(n+1)(n+2)js_{l+1}t_{l+1}\dots s_{\frac{n}{2}}t_{\frac{n}{2}}\big),\\&b_{s_1t_1}\dots b_{s_{l-1}t_{l-1}}b_{i(n+1)}b_{(n+2)j}b_{s_{l+1}t_{l+1}}\dots b_{s_{\frac{n}{2}}}b_{\frac{n}{2}}\\&\quad=a_{s_1t_1}\dots a_{s_{l-1}t_{l-1}}\sqrt{a_{ij}}\sqrt{a_{ij}}a_{s_{l+1}t_{l+1}}\dots a_{s_{\frac{n}{2}}t_{\frac{n}{2}}}\\&\quad=a_{s_1t_1}\dots a_{s_{l-1}t_{l-1}}a_{ij}a_{s_{l+1}t_{l+1}}\dots a_{s_{\frac{n}{2}}t_{\frac{n}{2}}}.\end{aligned}$$

Thus we have shown that if $\pi\in\mathcal{M}_1$ then we have $b_\pi=b_{\overline{\pi}}$. Similarly, we can prove that if $\pi\in\mathcal{M}_2$ then we have $b_\pi=b_{\overline{\pi}}$. So we have proved that $Pf(A(G^e))=Pf(A(\overline{G}^e))$, and the lemma follows. □

The above lemma can be proved by expanding the Pfaffian $Pf(A(\overline{G}^e))$ along the last row of $A(\overline{G}^e)$. But the idea in our proof is useful for the proofs of our main results.

3. New Pfaffian identities

We first introduce some notation. In this section, we assume that $A=(a_{ij})_{n\times n}$ is a skew symmetric matrix with n even. Suppose $E=\{(i_l,j_l)\mid l=1,2,\dots,k\}$ is a subset of $[n]\times[n]$

such that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ and $i_l < j_l$ for $1 \leqslant l \leqslant k$. We define a new skew symmetric matrix $E(A)$ of order n from A and E as follows:

$$E(A) = (b_{ij})_{n\times n}, \quad b_{ij} = \begin{cases} a_{ij} & \text{if } (i,j) \notin E \text{ and } i < j, \\ -a_{ji} & \text{if } (j,i) \notin E \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of $E(A)$, it is obtained from A by replacing all (i_l, j_l)- and (j_l, i_l)-entries with *zeros* and not changing the other entries, and hence it is a skew symmetric matrix. For example, if $A = (a_{ij})_{4\times 4}$ is a skew symmetric matrix and $E = \{(1,4), (2,3), (3,4)\}$, then

$$E(A) = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ -a_{12} & 0 & 0 & a_{24} \\ -a_{13} & 0 & 0 & 0 \\ 0 & -a_{24} & 0 & 0 \end{pmatrix}.$$

Now, we can state one of our main results as follows.

Theorem 3.1. *Suppose $A = (a_{ij})_{n\times n}$ is a skew symmetric matrix of order n and $E = \{(i_l, j_l) \mid l = 1, 2, \ldots, k\}$ is a nonempty subset of $[n] \times [n]$ such that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$, $i_l < j_l$ for $l \in [k]$. Then, for any fixed $p \in [k]$, we have*

$$\begin{aligned} &Pf\big(E(A)\big)Pf(A) \\ &\quad = Pf\big(E_p(A)\big)Pf\big(\overline{E_p}(A)\big) + a_{i_p j_p} \sum_{1\leqslant l\leqslant k, l\neq p} a_{i_l j_l}\big[f(p,l)Pf\big(E(A)_{\{i_p,j_l\}}\big)Pf(A_{\{j_p,i_l\}}) \\ &\qquad - g(p,l)Pf\big(E(A)_{\{i_p,i_l\}}\big)Pf(A_{\{j_p,j_l\}})\big], \end{aligned}$$

where $E_p = E \setminus \{(i_p, j_p)\}$, $\overline{E_p} = \{(i_p, j_p)\}$, $f(p,l) = s([n], i_p j_l)s([n], j_p i_l)$, and $g(p,l) = s([n], i_p i_l)s([n], j_p j_l)$.

Proof. Let G^e be the corresponding directed graph of A defined as above, whose vertex set is $[n]$. Let $\overline{G}^e$ be the directed graph with the vertex set $[n+2k]$ obtained from G^e by replacing each arc between every pair of vertices i_l and j_l with three arcs $(i_l, n+2l-1)$, $(n+2l-1, n+2l)$, and $(n+2l, j_l)$ with weights $\sqrt{a_{i_l j_l}}$, 1, and $\sqrt{a_{i_l j_l}}$ if (i_l, j_l) is an arc of G^e and with three arcs $(j_l, n+2l-1)$, $(n+2l-1, n+2l)$, and $(n+2l, i_l)$ with weights $\sqrt{a_{j_l i_l}}$, 1, and $\sqrt{a_{j_l i_l}}$ if (j_l, i_l) is an arc of G^e, respectively. For the case $a_{i_l j_l} > 0$ for $1 \leqslant l \leqslant k$, Fig. 2(a) and (b) illustrate the procedure constructing $\overline{G}^e$ from G^e. Suppose $\overline{A} = A(\overline{G}^e)$ is the skew adjacency matrix of $\overline{G}^e$.

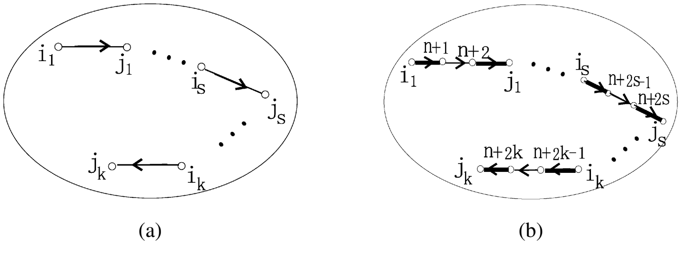

Fig. 2. (a) The directed graph G^e. (b) The directed graph $\overline{G}^e$.

Take $\alpha = [n]$, $\beta = \{n+1, n+2, \ldots, n+2k\} = \{x_i \mid x_i = n+i, 1 \leqslant i \leqslant 2k\}$. Take $q = 2p-1$. Hence $x_q = n + 2p - 1$ and $(-1)^{l+q+1} = (-1)^l$. By Lemma 2.2, we have

$$Pf_{\overline{A}}(\alpha)Pf_{\overline{A}}(\alpha\beta) = \sum_{l=1}^{2k}(-1)^l Pf_{\overline{A}}(\alpha x_q x_l)Pf_{\overline{A}}(\alpha\beta \setminus x_q x_l). \tag{3.1}$$

By the definitions of $E(A)$ and G^e and Lemma 2.4, we have

$$Pf_{\overline{A}}(\alpha) = Pf\big(E(A)\big), \qquad Pf_{\overline{A}}(\alpha\beta) = Pf(A). \tag{3.2}$$

We set

$$\begin{aligned} a_{l'} &= -Pf_{\overline{A}}(\alpha x_q x_{2l'-1})Pf_{\overline{A}}(\alpha\beta \setminus x_q x_{2l'-1}),\\ b_{l'} &= Pf_{\overline{A}}(\alpha x_q x_{2l'})Pf_{\overline{A}}(\alpha\beta \setminus x_q x_{2l'}). \end{aligned}$$

That is,

$$a_{l'} = -Pf_{\overline{A}}\big(\alpha(n+2p-1)(n+2l'-1)\big)Pf_{\overline{A}}\big(\alpha\beta \setminus (n+2p-1)(n+2l'-1)\big), \tag{3.3}$$

$$b_{l'} = Pf_{\overline{A}}\big(\alpha(n+2p-1)(n+2l')\big)Pf_{\overline{A}}\big(\alpha\beta \setminus (n+2p-1)(n+2l')\big). \tag{3.4}$$

By Lemma 2.4, it is not difficult to see that

$$b_p = Pf_{\overline{A}}(\alpha x_q x_p)Pf_{\overline{A}}\big(\alpha\beta \setminus \{x_q x_p\}\big) = Pf\big(E_p(A)\big)Pf\big(\overline{E_p}(A)\big). \tag{3.5}$$

Note that $a_p = 0$. Hence we have

$$\begin{aligned} Pf_{\overline{A}}(\alpha)Pf_{\overline{A}}(\alpha\beta) &= \sum_{l=1}^{2k}(-1)^l Pf_{\overline{A}}(\alpha x_q x_l)Pf_{\overline{A}}(\alpha\beta \setminus x_q x_l)\\ &= Pf\big(E_p(A)\big)Pf\big(\overline{E_p}(A)\big)' + \sum_{1 \leqslant l' \leqslant k, l' \neq p}(a_{l'} + b_{l'}). \end{aligned} \tag{3.6}$$

Obviously, if $a_{i_p j_p} = 0$ then the theorem is trivial. Hence we may assume that $a_{i_p j_p} \neq 0$.

First, we prove that if $a_{i_p j_p} > 0$ then the theorem holds. From (3.2) and (3.6) it suffices to prove the following claim:

Claim. *For any $l' \in [k]$ and $l' \neq p$, if $a_{i_p j_p} > 0$ we have*

$$\begin{aligned} a_{l'} + b_{l'} = {} & s\big(\big([n]\big), i_p j_{l'}\big)s\big([n], j_p i_{l'}\big)a_{i_p j_p}a_{i_{l'} j_{l'}}Pf\big(E(A)_{\{i_p, j_{l'}\}}\big)Pf(A_{\{j_p, i_{l'}\}})\\ & - s\big([n], i_p i_{l'}\big)s\big([n], j_p j_{l'}\big)a_{i_p j_p}a_{i_{l'} j_{l'}}Pf\big(E(A)_{\{i_p, i_{l'}\}}\big)Pf(A_{\{j_p, j_{l'}\}}). \end{aligned} \tag{3.7}$$

Suppose $a_{i_p j_p} > 0$. Then $(i_p, n+2p-1)$, $(n+2p-1, n+2p)$, and $(n+2p, j_p)$ are three arcs of $\overline{G}^e$ with weights $\sqrt{a_{i_p j_p}}$, 1, and $\sqrt{a_{i_p j_p}}$. We need to consider two cases:

(a) $a_{i_{l'} j_{l'}} \geqslant 0$;
(b) $a_{i_{l'} j_{l'}} < 0$.

If $a_{i_{l'} j_{l'}} \geqslant 0$, then $(i_{l'}, n+2l'-1)$, $(n+2l'-1, n+2l')$, and $(n+2l', j_{l'})$ are three arcs of $\overline{G}^e$ with weights $\sqrt{a_{i_{l'} j_{l'}}}$, 1, and $\sqrt{a_{i_{l'} j_{l'}}}$. Suppose X is a subset of the vertex set of $\overline{G}$. Let $\overline{G}[X]^e$ be the directed subgraph of $\overline{G}^e$ induced by X and $\overline{G}[X]$ the underlying graph of $\overline{G}[X]^e$. Note that $\overline{G}[\alpha(n+2l'-1)(n+2p-1)]$ contains two pendant edges $(i_{l'}, n+2i_{l'}-1)$ and $(i_p, n+2p-1)$. Each perfect matching π of $\overline{G}[\alpha(n+2l'-1)(n+2p-1)]$ can be denoted by $\pi = \pi' \cup \{(i_{l'}, n+2i_{l'}-1), (i_p, n+2p-1)\}$, where π' is a perfect matching of $\overline{G}[\alpha \setminus \{i_{l'}, i_p\}]$. Set

$$Pf\big(A\big(\overline{G}\big[\alpha(n+2l'-1)(n+2p-1)\big]^e\big)\big) = \sum_{\pi \in \mathcal{M}(\overline{G}[\alpha(n+2l'-1)(n+2p-1)])} b_\pi,$$

$$Pf\big(A\big(\overline{G}\big[\alpha \setminus \{i_{l'}, i_p\}\big]^e\big)\big) = \sum_{\pi' \in \mathcal{M}(\overline{G}[\alpha \setminus \{i_{l'}, i_p\}])} b_{\pi'},$$

where $\mathcal{M}(G)$ is the set of perfect matchings of a graph G. By the definitions of b_π and $b_{\pi'}$, it is not difficult to see that

$$b_\pi = \operatorname{sgn}(p-l')s\big([n], i_p i_{l'}\big)\sqrt{a_{i_p j_p} a_{i_{l'} j_{l'}}}\, b_{\pi'},$$

where $\operatorname{sgn}(x)$ denotes the sign of x. By the definition of $E(A)$, we have

$$Pf\big(A\big(\overline{G}\big[\alpha \setminus \{i_{l'}, i_p\}\big]^e\big)\big) = Pf\big(E(A)_{\{i_{l'}, i_p\}}\big).$$

Hence we have proved the following:

$$Pf_{\overline{A}}\big(\alpha(n+2l'-1)(n+2p-1)\big) = \operatorname{sgn}(p-l')s\big([n], i_p i_{l'}\big)\sqrt{a_{i_p j_p} a_{i_{l'} j_{l'}}}\, Pf\big(E(A)_{\{i_p, i_{l'}\}}\big). \quad (3.8)$$

Similarly, we can prove the following:

$$Pf_{\overline{A}}\big(\alpha\beta \setminus (n+2l'-1)(n+2p-1)\big) = \operatorname{sgn}(p-l')s\big([n], j_p j_{l'}\big)\sqrt{a_{i_p j_p} a_{i_{l'} j_{l'}}}\, Pf(A_{\{j_p, j_{l'}\}}); \quad (3.9)$$

$$Pf_{\overline{A}}\big(\alpha(n+2l')(n+2p-1)\big) = \operatorname{sgn}(l'-p)s\big([n], i_p j_{l'}\big)\sqrt{a_{i_p j_p} a_{i_{l'} j_{l'}}}\, Pf\big(E(A)_{\{i_p, j_{l'}\}}\big); \quad (3.10)$$

$$Pf_{\overline{A}}\big(\alpha\beta \setminus (n+2l')(n+2p-1)\big) = \operatorname{sgn}(l'-p)s\big([n], j_p i_{l'}\big)\sqrt{a_{i_p j_p} a_{i_{l'} j_{l'}}}\, Pf(A_{\{j_p, i_{l'}\}}). \quad (3.11)$$

Then (3.7) is immediate from (3.3), (3.4), (3.8)–(3.11). Hence if $a_{i_{l'} j_{l'}} \geqslant 0$ then the claim follows.

If $a_{i_{l'} j_{l'}} < 0$, then $(j_{l'}, n+2l'-1)$, $(n+2l'-1, n+2l')$, and $(n+2l', i_{l'})$ are three arcs of $\overline{G}^e$ with weights $\sqrt{-a_{i_{l'} j_{l'}}}$, 1, and $\sqrt{-a_{i_{l'} j_{l'}}}$. Similarly, we can prove the following:

$$\begin{aligned} &Pf_{\overline{A}}\big(\alpha(n+2l'-1)(n+2p-1)\big) \\ &\quad = \operatorname{sgn}(p-l')s\big([n], i_p j_{l'}\big)\sqrt{a_{i_p j_p} a_{j_{l'} i_{l'}}}\, Pf\big(A\big(E(A)_{\{i_p, j_{l'}\}}\big)\big); \end{aligned} \quad (3.12)$$

$$\begin{aligned} &Pf_{\overline{A}}\big(\alpha\beta \setminus (n+2l'-1)(n+2p-1)\big) \\ &\quad = \operatorname{sgn}(p-l')s\big([n], j_p i_{l'}\big)\sqrt{a_{i_p j_p} a_{j_{l'} i_{l'}}}\, Pf(A_{\{j_p, i_{l'}\}}); \end{aligned} \quad (3.13)$$

$$Pf_{\overline{A}}\big(\alpha(n+2l')(n+2p-1)\big) = \operatorname{sgn}(l'-p)s\big([n], i_p i_{l'}\big)\sqrt{a_{i_p j_p} a_{j_{l'} i_{l'}}} Pf\big(E(A)_{\{i_p, i_{l'}\}}\big); \quad (3.14)$$

$$Pf_{\overline{A}}\big(\alpha\beta \setminus (n+2l')(n+2p-1)\big) = \operatorname{sgn}(l'-p)s\big([n], j_p j_{l'}\big)\sqrt{a_{i_p j_p} a_{j_{l'} i_{l'}}} Pf(A_{\{j_p, j_{l'}\}}). \quad (3.15)$$

Then (3.7) is immediate from (3.3), (3.4), (3.12)–(3.15). Hence if $a_{i_{l'} j_{l'}} < 0$ then the claim follows.

Hence we have proved that if $a_{i_p j_p} > 0$ then the theorem holds.

If $a_{i_p j_p} < 0$, we consider $Pf(-A)$ and $Pf(-E(A))$. Note that $(-A)_{i_p j_p} > 0$. The result proved above implies that

$$\begin{aligned} &Pf\big(-E(A)\big)Pf(-A) \\ &\quad = Pf\big(E_p(-A)\big)Pf\big(\overline{E_p}(-A)\big) - a_{i_p j_p} \sum_{1 \leqslant l \leqslant k,\, l \neq p} (-a_{i_l j_l}) \\ &\qquad \times \big[f(p,l) \times Pf\big(E(-A)_{\{i_p, j_l\}}\big)Pf\big((-A)_{\{j_p, i_l\}}\big) \\ &\qquad - g(p,l)Pf\big(E(-A)_{\{i_p, i_l\}}\big)Pf\big((-A)_{\{j_p, j_l\}}\big)\big]. \end{aligned} \quad (3.16)$$

Note that by the definition of the Pfaffian we have $Pf(-A) = (-1)^{\frac{n}{2}} Pf(A)$. By (3.16), we can show that we have

$$\begin{aligned} &Pf\big(E(A)\big)Pf(A) \\ &\quad = Pf\big(E_p(A)\big)Pf\big(\overline{E_p}(A)\big) + a_{i_p j_p} \sum_{1 \leqslant l \leqslant k,\, l \neq p} a_{i_l j_l} \\ &\qquad \times \big[f(p,l)Pf\big(E(A)_{\{i_p, j_l\}}\big)Pf(A_{\{j_p, i_l\}}) - g(p,l)Pf\big(E(A)_{\{i_p, i_l\}}\big)Pf(A_{\{j_p, j_l\}})\big] \end{aligned}$$

implying that if $a_{i_p j_p} < 0$ then the theorem also holds.

Hence we have proved the theorem. □

Corollary 3.2. *With the same notation as Theorem* 3.1*, for any fixed* $p \in [k]$*,*

$$\begin{aligned} &Pf\big(E(A)\big)Pf(A) \\ &\quad = Pf\big(E_p(A)\big)Pf\big(\overline{E_p}(A)\big) + a_{i_p j_p} \sum_{1 \leqslant l \leqslant k,\, l \neq p} a_{i_l j_l} \\ &\qquad \times \big[f(p,l)Pf\big(E(A)_{\{j_p, i_l\}}\big)Pf(A_{\{i_p, j_l\}}) - g(p,l)Pf\big(E(A)_{\{j_p, j_l\}}\big)Pf(A_{\{i_p, i_l\}})\big]. \end{aligned}$$

Proof. Let A^T be the transpose of A. Note that $Pf(A^T) = (-1)^{\frac{n}{2}} Pf(A)$. The corollary follows immediately from Theorem 3.1 by considering the transpose of A. □

The following result is a special case of Theorem 3.1 and Corollary 3.2.

Corollary 3.3. *Suppose* $A = (a_{ij})_{n \times n}$ *is a skew symmetric matrix of order* n *and* $E = \{(i_l, j_l) \mid l = 1, 2, \ldots, k\}$ *is a nonempty subset of* $[n] \times [n]$ *such that* $i_1 < j_1 < i_2 < j_2 < \cdots < i_l < j_l < \cdots < i_k < j_k$. *Then*

$$Pf\big(E(A)\big)Pf(A) - Pf\big(E_1(A)\big)Pf\big(\overline{E_1}(A)\big)$$
$$= a_{i_1 j_1} \sum_{l=2}^{k} a_{i_l j_l} \big[Pf\big(E(A)_{\{i_1, j_l\}}\big)Pf(A_{\{j_1, i_l\}}) - Pf\big(E(A)_{\{i_1, i_l\}}\big)Pf(A_{\{j_1, j_l\}})\big]$$
$$= a_{i_1 j_1} \sum_{l=2}^{k} a_{i_l j_l} \big[Pf\big(E(A)_{\{j_1, i_l\}}\big)Pf(A_{\{i_1, j_l\}}) - Pf\big(E(A)_{\{j_1, j_l\}}\big)Pf(A_{\{i_1, i_l\}})\big].$$

The Pfaffian identities in Theorem 3.1 and Corollaries 3.2 and 3.3 express the product of Pfaffians of two skew symmetric matrices $E(A)$ and A in terms of the Pfaffians of the minors of $E(A)$ and A, where $E(A)$ is a skew symmetric matrix obtained from A by replacing some nonzero entries $a_{i_l j_l}$ and $a_{j_l i_l}$ of A with zeros. On the other hand, an obvious observation in the Pfaffian identities known before, which belong to the Plücker relations, is that the related matrices are either a skew symmetric matrix A or some minors of A. Hence the Pfaffian identities in Theorem 3.1 and Corollaries 3.2 and 3.3 are completely new and different from the Plücker relations.

Example 3.1. Let $A = (a_{ij})_{4\times 4}$ and $E = \{(1,2), (3,4)\}$. Then, by Corollary 3.3, we have

$$Pf\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} Pf\begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & 0 \\ -a_{14} & -a_{24} & 0 & 0 \end{pmatrix}$$
$$= Pf\begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} Pf\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & 0 \\ -a_{14} & -a_{24} & 0 & 0 \end{pmatrix}$$
$$+ a_{12}a_{34} Pf\begin{pmatrix} 0 & a_{23} \\ -a_{23} & 0 \end{pmatrix} Pf\begin{pmatrix} 0 & a_{14} \\ -a_{14} & 0 \end{pmatrix}$$
$$- a_{12}a_{34} Pf\begin{pmatrix} 0 & a_{24} \\ -a_{24} & 0 \end{pmatrix} Pf\begin{pmatrix} 0 & a_{13} \\ -a_{13} & 0 \end{pmatrix}.$$

4. Applications

As applications of some results in Section 3, we obtain some new determinant identities related to the Plücker relations in Section 4.1, and we prove a quadratic relation for the number of perfect matchings of plane graphs in Section 4.2, which has a simpler form than the formula in [34].

4.1. New determinant identities

We first introduce some notation and terminology. Throughout this subsection, we will assume $A = (a_{ij})_{n\times n}$ is an arbitrary matrix of order n, and $E = \{(i_l, j_l) \mid 1 \leqslant l \leqslant k\} \subseteq [n] \times [n]$, where

$a_{i_l j_l} \neq 0$. Define a new matrix of order n from A and E, denoted by $E[A] = (b_{st})_{n\times n}$, where

$$b_{st} = \begin{cases} a_{st} & \text{if } (s,t) \notin E, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $E[A]$ is an $n \times n$ matrix obtained from A by replacing all entries $a_{i_l j_l}$ for $1 \leqslant l \leqslant k$ with zeros and not changing the other entries. For example, if $A = (a_{ij})_{4\times 4}$, $E = \{(1,2),(2,2),(3,1)\}$, by the definition of $E[A]$ we have

$$E[A] = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \qquad \bigl\{(3,4)\bigr\}[A] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

We need the following lemma:

Lemma 4.5. *If $A = (a_{ij})_{n\times n}$ is a matrix of order n and $A^* = \left(\begin{smallmatrix} 0 & A \\ -A^T & 0 \end{smallmatrix}\right)$, then, for any $i, j \in [n]$, $i \neq j$, we have*

(i) $Pf(A^*_{\{i,j\}}) = 0$,
(ii) $Pf(A^*_{\{i,n+j\}}) = (-1)^{\frac{1}{2}(n-1)(n-2)} \det(A_{ij})$,
(iii) $Pf(A^*_{\{n+i,j\}}) = (-1)^{\frac{1}{2}(n-1)(n-2)} \det(A_{ji})$,

where A_{ij} denotes the minor of A obtained by deleting the ith row and jth column from A.

Proof. Note that $A^*_{\{i,j\}} = \left(\begin{smallmatrix} 0 & B \\ -B^T & 0 \end{smallmatrix}\right)$, where B is an $(n-2)\times n$ matrix obtained from A by deleting two rows indexed by i and j. Obviously, $\det\left(\begin{smallmatrix} 0 & B \\ -B^T & 0 \end{smallmatrix}\right) = 0$. Hence by Cayley's Theorem we have $[Pf(A^*_{\{i,j\}})]^2 = \det\left(\begin{smallmatrix} 0 & B \\ -B^T & 0 \end{smallmatrix}\right) = 0$, which implies that $Pf(A^*_{\{i,j\}}) = 0$. Similarly, by Lemma 2.3 we can prove (ii) and (iii). Hence the lemma follows. □

Theorem 4.2. *Let $A = (a_{ij})_{n\times n}$ be a matrix of order n and $E = \{(i_l, j_l) \mid 1 \leqslant l \leqslant k\}$ a nonempty subset of $[n]\times[n]$, where $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$. Then for a fixed $p \in [k]$ we have*

$$\begin{aligned} &\det\bigl(E[A]\bigr)\det(A) \\ &\quad = \det\bigl(E_p[A]\bigr)\det\bigl(\overline{E_p}[A]\bigr) - \sum_{1\leqslant l\leqslant k,\, l\neq p} (-1)^{i_l+j_l+i_p+j_p} a_{i_p j_p} a_{i_l j_l} \det\bigl(E[A]_{i_p j_l}\bigr)\det(A_{i_l j_p}), \end{aligned}$$

where $E_p = E \setminus \{(i_p, j_p)\}$ and $\overline{E_p} = \{(i_p, j_p)\}$.

Proof. Define

$$A^* = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} = (a^*_{ij})_{2n\times 2n} \quad \text{and} \quad E^* = \bigl\{(i_l, n+j_l) \mid 1 \leqslant l \leqslant k\bigr\}.$$

By Theorem 3.1, we have

$$Pf\big(E^*(A^*)\big)Pf(A^*) = Pf\big(E_p^*(A^*)\big)Pf\big(\overline{E_p^*}(A^*)\big) + a^*_{i_p(n+j_p)} \sum_{1\leqslant l\leqslant k,\, l\neq p} a^*_{i_l(n+j_l)}\big[f(p,l)Pf\big(E^*(A^*)_{\{i_p,n+j_l\}}\big)Pf\big(A^*_{\{n+j_p,i_l\}}\big) - g(p,l)Pf\big(E^*(A^*)_{\{i_p,i_l\}}\big)Pf\big(A^*_{\{n+j_p,n+j_l\}}\big)\big], \tag{4.1}$$

where $f(p,l) = s([2n], i_p(n+j_l))s([2n], (n+j_p)i_l)$ and $g(p,l) = s([2n], i_p i_l)s([2n], (n+j_p)(n+j_l))$. It is not difficult to see that we have the following:

$$a^*_{i_p(n+j_p)} = a_{i_p j_p}, \qquad a^*_{i_l(n+j_l)} = a_{i_l j_l}, \qquad f(p,l) = -(-1)^{i_p+j_p+i_l+j_l}. \tag{4.2}$$

By Lemma 2.3 and the definitions of A^* and $E^*(A^*)$, we have

$$Pf\big(E^*(A^*)\big) = (-1)^{\frac{1}{2}n(n-1)}\det\big(E[A]\big), \qquad Pf(A^*) = (-1)^{\frac{1}{2}n(n-1)}\det(A), \tag{4.3}$$

$$Pf\big(E_p^*(A^*)\big) = (-1)^{\frac{1}{2}n(n-1)}\det\big(E_p[A]\big), \qquad Pf\big(\overline{E_p^*}(A^*)\big) = (-1)^{\frac{1}{2}n(n-1)}\det\big(\overline{E_p}[A]\big). \tag{4.4}$$

By (i) in Lemma 4.5, we have

$$Pf\big(E^*(A^*)_{\{i_p,i_l\}}\big) = 0, \tag{4.5}$$

and by (ii) and (iii) in Lemma 4.5, we have

$$Pf\big(E^*(A^*)_{\{i_p,n+j_l\}}\big) = (-1)^{\frac{1}{2}(n-1)(n-2)}\det\big(E[A]_{i_p j_l}\big), \tag{4.6}$$

$$Pf\big(A^*_{\{n+j_p,i_l\}}\big) = (-1)^{\frac{1}{2}(n-1)(n-2)}\det(A_{i_l j_p}). \tag{4.7}$$

The theorem is immediate from (4.1)–(4.7), and hence we have completed the proof of the theorem. □

In the proof of Theorem 4.2, (4.1) is obtained from Theorem 3.1. Obviously, an identity similar to (4.1) can be obtained from Corollary 3.2. By this identity we can prove the following:

Theorem 4.3. *Let $A = (a_{ij})_{n\times n}$ be a matrix of order n and $E = \{(i_l, j_l) \mid 1 \leqslant l \leqslant k\}$ a nonempty subset of $[n]\times[n]$, where $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$. Then for a fixed $p \in [k]$ we have*

$$\det\big(E[A]\big)\det(A) = \det\big(E_p[A]\big)\det\big(\overline{E_p}[A]\big) - \sum_{1\leqslant l\leqslant k,\, l\neq p} (-1)^{i_l+j_l+i_p+j_p} a_{i_p j_p} a_{i_l j_l} \det\big(E[A]_{i_l j_p}\big)\det(A_{i_p j_l}),$$

where $E_p = E \setminus \{(i_p, j_p)\}$ and $\overline{E_p} = \{(i_p, j_p)\}$.

The following result is immediate from Theorems 4.2 and 4.3.

Corollary 4.4. *Let $A = (a_{ij})_{n\times n}$ be a matrix of order n and $E = \{(i_l, j_l) \mid 1 \leqslant l \leqslant k\}$ a nonempty subset of $[n] \times [n]$, where $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$. Then for a fixed $p \in [k]$ we have*

$$\sum_{l=1}^{k}(-1)^{i_l+j_l} a_{i_l j_l}\left\{\det\left(A[E]_{i_p j_l}\right)\det(A_{i_l j_p}) - \det\left(E[A]_{i_l j_p}\right)\det(A_{i_p j_l})\right\} = 0.$$

Example 4.2. Let $A = (a_{ij})_{3\times 3}$, $E = \{(1,1), (2,2), (3,3)\}$, and $p = 2$. Then, by Theorems 4.2 and 4.3, we have

$$\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix} \\
&\quad = -a_{11}a_{22}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & 0 \end{vmatrix} - a_{22}a_{33}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} 0 & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
&\quad = -a_{11}a_{22}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & 0 \end{vmatrix} - a_{22}a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} 0 & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.
\end{aligned}$$

4.2. Graphical edge condensation for enumerating perfect matchings

Let $M(G)$ denote the sum of weights of perfect matchings of a weighted graph G, where the weight of a perfect matching M of G is defined as the product of weights of edges in M. It is well known that computing $M(G)$ of a graph G is an *NP*-complete problem (see [10]). Inspired by (2.2), Dodgson's determinant-evaluation rule, Propp [26] first found the method of graphical vertex condensation for enumerating perfect matchings of plane bipartite graphs as follows:

Proposition 4.1. *(See Propp [26].) Let $G = (U, V)$ be a plane bipartite graph in which $|U| = |V|$. Let vertices a, b, c, and d form a 4-cycle face in G, $a, c \in U$, and $b, d \in V$. Then*

$$M(G)M\left(G - \{a,b,c,d\}\right) = M\left(G - \{a,b\}\right)M\left(G - \{c,d\}\right) + M\left(G - \{a,d\}\right)M\left(G - \{b,c\}\right).$$

By a combinatorial method, Kuo [14] generalized Propp's result above as follows.

Proposition 4.2. *(See Kuo [14].) Let $G = (U, V)$ be a plane bipartite graph in which $|U| = |V|$. Let vertices a, b, c, and d appear in a cyclic order on a face of G.*

(1) *If $a, c \in U$, and $b, d \in V$, then*

$$\begin{aligned}&M(G)M\left(G - \{a,b,c,d\}\right) \\ &\quad = M\left(G - \{a,b\}\right)M\left(G - \{c,d\}\right) + M\left(G - \{a,d\}\right)M\left(G - \{b,c\}\right).\end{aligned}$$

(2) *If $a, b \in U$, and $c, d \in V$, then*

$$\begin{aligned}&M(G)M\left(G - \{a,b,c,d\}\right) \\ &\quad = M\left(G - \{a,d\}\right)M\left(G - \{b,c\}\right) - M\left(G - \{a,c\}\right)M\left(G - \{b,d\}\right).\end{aligned}$$

Using Ciucu's Matching Factorization Theorem in [2], Yan and Zhang [36] obtained a more general result than Kuo's for the method of graphical vertex condensation for enumerating perfect matchings of plane bipartite graphs. Furthermore, Yan et al. [34] proved the following results:

Proposition 4.3. *(See Yan, Yeh, and Zhang [34].) Let G be a plane weighted graph with $2n$ vertices. Let vertices $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ $(2 \leqslant k \leqslant n)$ appear in a cyclic order on a face of G, and let $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$. Then, for any $j = 1, 2, \dots, k$, we have*

$$\sum_{Y \subseteq B,\, |Y| \text{ is odd}} M(G - a_j - Y) M\big(G - A \setminus \{a_j\} - \overline{Y}\big) = \sum_{W \subseteq B,\, |W| \text{ is even}} M(G - W) M(G - A - \overline{W}),$$

where the first sum ranges over all odd subsets Y of B, the second sum ranges over all even subsets W of B, $\overline{Y} = B \setminus Y$, and $\overline{W} = B \setminus W$.

The following result, which is a special case of the above theorem, was first found by Kenyon and was sent to "Domino Forum" in an Email (for details, see [34]).

Corollary 4.5. *Let G be a plane graph with four vertices a, b, c, and d (in the cyclic order) adjacent to a single face. Then*

$$\begin{aligned} &M(G)M(G-a-b-c-d) + M(G-a-c)M(G-b-d) \\ &\quad = M(G-a-b)M(G-c-d) + M(G-a-d)M(G-b-c). \end{aligned} \tag{4.8}$$

Using Ciucu's Matching Factorization Theorem, Yan et al. [34] also obtained some results for the method of graphical edge condensation for enumerating perfect matchings of plane graphs. In this subsection, by using the new Pfaffian identity from Corollary 3.3 we will prove a quadratic relation which has a simpler form than the formula in [34], for the method of graphical edge condensation for computing perfect matchings of plane graphs.

We first introduce the Pfaffian method for enumerating perfect matchings [11,12]. If G^e is an orientation of a simple graph G and C is a cycle of even length, we say that C is oddly oriented in G^e if C contains odd number of edges that are directed in G^e in the direction of each orientation of C. We say that G^e is a Pfaffian orientation of G if every nice cycle of even length of G is oddly oriented in G^e (a cycle C in G is nice if $G - C$ has perfect matchings). It is well known that if a graph G contains no subdivision of $K_{3,3}$, then G has a Pfaffian orientation (see [17]). McCuaig [20], McCuaig et al. [21], and Robertson et al. [27] found a polynomial-time algorithm to show whether a bipartite graph has a Pfaffian orientation. For some related recent papers, see for example [35,37].

Proposition 4.4. *(See [12,18].) Let G^e be a Pfaffian orientation of a graph G. Then*

$$\big[M(G)\big]^2 = \det\big(A(G^e)\big),$$

where $A(G^e)$ is the skew adjacency matrix of G^e.

Let G^e be a Pfaffian orientation of a graph G and $A(G^e)$ the skew adjacency matrix of G^e. By Cayley's Theorem and Proposition 4.4, we have

$$M(G) = \pm Pf\big(A\big(G^e\big)\big),$$

which implies that, for two arbitrary perfect matchings π_1 and π_2 of G, both b_{π_1} and b_{π_2} have the same sign.

Proposition 4.5. *(See Kasteleyn's Theorem [11,12,18].) Every plane graph G has an orientation G^e such that every boundary face, except possibly the unbounded face, has an odd number of edges oriented clockwise. Furthermore, such an orientation is a Pfaffian orientation.*

Now we can prove the following result:

Lemma 4.6. *Let G be a plane graph with four vertices a, b, c, and d (in a cyclic order) adjacent to the unbounded face. Let G^e be an arbitrary Pfaffian orientation satisfying the condition in Proposition 4.5, and let $A = A(G^e)$ be the skew adjacency matrix of G^e. Then all of $Pf(A_{\{a,b,c,d\}})Pf(A)$, $Pf(A_{\{a,b\}})Pf(A_{\{c,d\}})$, $Pf(A_{\{a,c\}})Pf(A_{\{b,d\}})$, and $Pf(A_{\{a,d\}})Pf(A_{\{b,c\}})$ have the same sign.*

Proof. By (2.1) in Corollary 2.1, we have

$$\begin{aligned} &Pf(A_{\{a,b,c,d\}})Pf(A) \\ &\quad = Pf(A_{\{a,b\}})Pf(A_{\{c,d\}}) - Pf(A_{\{a,c\}})Pf(A_{\{b,d\}}) + Pf(A_{\{a,d\}})Pf(A_{\{b,c\}}). \end{aligned} \tag{4.9}$$

Obviously, $A_{\{a,b,c,d\}}$, $A_{\{a,b\}}$, $A_{\{c,d\}}$, $A_{\{a,c\}}$, $A_{\{b,d\}}$, $A_{\{a,d\}}$, and $A_{\{b,c\}}$ are the skew adjacency matrices of $G^e - a - b - c - d$, $G^e - a - b$, $G^e - c - d$, $G^e - a - c$, $G^e - b - d$, $G^e - a - d$, and $G^e - b - c$, respectively. Note that all the orientations $G^e - a - b - c - d$, $G^e - a - b$, $G^e - c - d$, $G^e - a - c$, $G^e - b - d$, $G^e - a - d$, and $G^e - b - c$ of $G - a - b - c - d$, $G - a - b$, $G - c - d$, $G - a - c$, $G - b - d$, $G - a - d$, and $G - b - c$ satisfy the condition in Proposition 4.6, and hence are Pfaffian orientations. By the remarks of Proposition 4.4, we have

$$M(G) = \pm Pf(A), \qquad M(G - a - b - c - d) = \pm Pf(A_{\{a,b,c,d\}}).$$

Hence we have proved the following:

$$M(G)M(G - a - b - c - d) = \pm Pf(A)Pf(A_{\{a,b,c,d\}}). \tag{4.10}$$

Similarly, we can prove the following:

$$M(G - a - b)M(G - c - d) = \pm Pf(A_{\{a,b\}})Pf(A_{\{c,d\}}); \tag{4.11}$$

$$M(G - a - c)M(G - b - d) = \pm Pf(A_{\{a,c\}})Pf(A_{\{b,d\}}); \tag{4.12}$$

$$M(G - a - d)M(G - b - c) = \pm Pf(A_{\{a,d\}})Pf(A_{\{b,c\}}). \tag{4.13}$$

The lemma is immediate from (4.8)–(4.13). □

We now state the main result of this subsection.

Theorem 4.4. *Suppose G is a plane weighted graph with an even number of vertices and denote the weight of every edge e in G by ω_e. Let $e_1 = a_1b_1$, $e_2 = a_2b_2, \ldots, e_k = a_kb_k$ $(k \geqslant 2)$ be k independent edges in the boundary of a face f of G, and let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on f, and let $X = \{e_i \mid i = 1, 2, \ldots, k\}$. Then, for any $j = 1, 2, \ldots, k$,*

$$\begin{aligned} M(G)M(G-X) = M(G-e_j)M\big(G-X\setminus\{e_j\}\big) \\ + \omega_{e_j} \sum_{1\leqslant i\leqslant k,\, i\neq j} \omega_{e_i}\big[M(G-b_j-a_i)M(G-X-a_j-b_i) \\ - M(G-b_j-b_i)M(G-X-a_j-a_i)\big]. \end{aligned}$$

Proof. Note that $e_1 = a_1b_1$, $e_2 = a_2b_2, \ldots, e_k = a_kb_k$ $(k \geqslant 2)$ are k independent edges in the boundary of a face f of G. It suffices to prove the following:

$$\begin{aligned} M(G)M(G-X) = M(G-e_1)M\big(G-X\setminus\{e_1\}\big) \\ + \omega_{e_1} \sum_{i=2}^{k} \omega_{e_i}\big[M(G-b_1-a_i)M(G-X-a_1-b_i) \\ - M(G-b_1-b_i)M(G-X-a_1-a_i)\big]. \end{aligned} \tag{4.14}$$

Since G is a plane graph, for an arbitrary face F of G there exists a planar embedding of G such that the face F is the unbounded one. Hence we may assume that vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on the unbounded face of G. Let T be a spanning trees containing the k edges e_i, $1 \leqslant i \leqslant k$, and let T^e be an orientation of T such that the direction of each edge e_i is from a_i to b_i for $i = 1, 2, \ldots, k$. Because each face of G can be obtained from T by adding edges, it is not difficult to see that there exists an orientation G^e of G obtained from T^e which satisfies the condition in Proposition 4.5. Hence all G^e, $G^e - X$, $G^e - e_j$, $G^e - X \setminus \{e_j\}$, $G^e - a_i - b_j$, $G^e - X - a_j - b_i$, $G^e - b_j - b_i$, and $G^e - X - a_i - a_j$ are Pfaffian orientations satisfying the condition in Proposition 4.5, the skew adjacency matrices of which are A, $E(A)$, $\overline{E_j}(A)$, $E_j(A)$, $A_{\{a_i,b_j\}}$, $E(A)_{\{a_j,b_i\}}$, $A_{\{b_j,b_i\}}$, and $E(A)_{\{a_i,a_j\}}$, respectively, where $E = \{(a_i, b_j) \mid 1 \leqslant i \leqslant k\}$, $E_j = E \setminus \{e_j\}$, and $\overline{E_j} = E \setminus E_j$. Without loss of generality, we may assume that $a_i = 2i - 1$ and $b_i = 2i$ for $i = 1, 2, \ldots, k$, that is, $E = \{(1, 2), (3, 4), \ldots, (2k - 1, 2k)\}$. By Corollary 3.3, we have

$$\begin{aligned} Pf\big(E(A)\big)Pf(A) = Pf\big(E_1(A)\big)Pf\big(\overline{E_1}(A)\big) \\ + a_{12} \sum_{i=2}^{k} a_{2i-1,2i}\big[Pf\big(E(A)_{\{1,2i\}}\big)Pf(A_{\{2,2i-1\}}) \\ - Pf\big(E(A)_{\{1,2i-1\}}\big)Pf(A_{\{2,2i\}})\big]. \end{aligned} \tag{4.15}$$

By a method similar to that in Lemma 4.6, we can prove that

$$Pf\big(E(A)\big)Pf(A) = \pm M(G-X)M(G); \tag{4.16}$$

$$Pf\big(E_1(A)\big)Pf\big(\overline{E_1}(A)\big) = \pm M\big(G-X\setminus\{e_1\}\big)M(G-e_1); \tag{4.17}$$

$$Pf\big(E(A)_{\{1,2i\}}\big)Pf(A_{\{2,2i-1\}}) = \pm M(G - X - a_1 - b_i)M(G - b_1 - a_i); \tag{4.18}$$

$$Pf\big(E(A)_{\{1,2i-1\}}\big)Pf(A_{\{2,2i\}}) = \pm M(G - X - a_1 - a_i)M(G - b_1 - b_i). \tag{4.19}$$

Since every perfect matching of $G - X$ is also a perfect matching of G, by the definition of the Pfaffian both $Pf(A)$ and $Pf(E(A))$ have the same sign. Hence by (4.16) we have

$$Pf\big(E(A)\big)Pf(A) = M(G - X)M(G). \tag{4.16$'$}$$

Similarly, we have

$$Pf\big(E_1(A)\big)Pf\big(\overline{E_1}(A)\big) = M\big(G - X \setminus \{e_1\}\big)M(G - e_1). \tag{4.17$'$}$$

Note that if π' is a perfect matching of $G - a_1 - b_1 - a_i - b_i$ $(i \neq 1)$, then $\pi = \pi' \cup \{(a_1, b_1), (a_i, b_i)\}$ is a perfect matching of G. By the definition of the Pfaffian, it is not difficult to see that both b_π and $b_{\pi'}$ have the same sign, which implies that both $Pf(A)$ and $Pf(A_{\{a_1,b_1,a_i,b_i\}})$ have the same sign. Hence $Pf(A)Pf(A_{\{a_1,b_1,a_i,b_i\}}) \geqslant 0$. By Lemma 4.6, we have

$$Pf(A_{\{a_1,b_i\}})Pf(A_{\{b_1,a_i\}}) \geqslant 0, \qquad Pf(A_{\{a_1,a_i\}})Pf(A_{\{b_1,b_i\}}) \geqslant 0. \tag{4.20}$$

Since every perfect matching of $G - X - a_1 - b_i$ is also a perfect matching of $G - a_1 - b_i$, both $Pf(E(A)_{\{a_1,b_i\}})$ and $Pf(A_{\{a_1,b_i\}})$ have the same sign. Similarly, both $Pf(E(A)_{\{a_1,a_i\}})$ and $Pf(A_{\{a_1,a_i\}})$ have the same sign. Hence by (4.20) we have

$$Pf\big(E(A)_{\{a_1,b_i\}}\big)Pf(A_{\{b_1,a_i\}}) \geqslant 0, \qquad Pf\big(E(A)_{\{a_1,a_i\}}\big)Pf(A_{\{b_1,b_i\}}) \geqslant 0. \tag{4.21}$$

From (4.18), (4.19) and (4.21), we have

$$Pf\big(E(A)_{\{1,2i\}}\big)Pf(A_{\{2,2i-1\}}) = M(G - X - a_1 - b_i)M(G - b_1 - a_i); \tag{4.18$'$}$$

$$Pf\big(E(A)_{\{1,2i-1\}}\big)Pf(A_{\{2,2i\}}) = M(G - X - a_1 - a_i)M(G - b_1 - b_i). \tag{4.19$'$}$$

Note that $a_{12} = \omega_{e_1}$ and $a_{2i-1,2i} = \omega_{e_i}$. It is not difficult to see that (4.14) follows from (4.15) and (4.16′)–(4.19′). Hence we have completed the proof of the theorem. □

The formula in Theorem 4.4 for the method of graphical edge condensation for enumerating perfect matchings of plane graphs has a simpler form than that in Theorem 3.2 in [34].

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References

[1] A. Cayley, Sur les déterminants gauches, J. Reine Angew. Math. 38 (1848) 93–96;
or Collected Mathematical Papers I, Cambridge Univ. Press, Cambridge, 1889–1897, pp. 410–413.

[2] M. Ciucu, Enumeration of perfect matchings in graphs with reflective symmetry, J. Combin. Theory Ser. A 77 (1997) 67–97.

[3] C.L. Dodgson, Condensation of determinants, Proc. R. Soc. Lond. 15 (1866) 150–155.

[4] A.W.M. Dress, W. Wenzel, A simple proof of an identity concerning Pfaffians of skew symmetric matrices, Adv. Math. 112 (1995) 120–134.

[5] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, New York, 1993.

[6] A.M. Hamel, Pfaffian identities: A combinatorial approach, J. Combin. Theory Ser. A 94 (2001) 205–217.

[7] R. Hirota, Soliton solutions to the BKP equations, I. The Pfaffian technique, J. Phys. Soc. Japan 58 (1989) 2285–2296.

[8] M. Ishikawa, M. Wakayama, Minor summation formulas of Pfaffians, Linear Multilinear Algebra 39 (1995) 285–305.

[9] M. Ishikawa, M. Wakayama, Applications of minor summation formula III, Plücker relations, lattice paths and Pfaffian identities, J. Combin. Theory Ser. A 113 (2006) 113–155.

[10] M. Jerrum, Two-dimensional monomer–dimer systems are computationally intractable, J. Stat. Phys. 48 (1987) 121–134.

[11] P.W. Kasteleyn, Dimer statistics and phase transition, J. Math. Phys. 4 (1963) 287–293.

[12] P.W. Kasteleyn, Graph theory and crystal physics, in: F. Harary (Ed.), Graph Theory and Theoretical Physics, Academic Press, 1967, pp. 43–110.

[13] D.E. Knuth, Overlapping Pfaffians, Electron. J. Combin. 3 (1996) R5.

[14] E.H. Kuo, Applications of graphical condensation for enumerating matchings and tilings, Theoret. Comput. Sci. 319 (2004) 29–57.

[15] E.H. Kuo, Graphical condensation generalizations involving Pfaffians and determinants, arXiv: math.CO/0605154.

[16] B. Leclerc, On identities satisfied by minors of a matrix, Adv. Math. 100 (1993) 101–132.

[17] C.H.C. Little, A characterization of convertible (0, 1)-matrices, J. Combin. Theory Ser. B 18 (1975) 187–208.

[18] L. Lovász, M. Plummer, Matching Theory, Ann. Discrete Math., vol. 29, North-Holland, New York, 1986.

[19] J.G. Luque, J.Y. Thibon, Pfaffian and Hafnian identities in shuffle algebras, Adv. in Appl. Math. 29 (2002) 620–646.

[20] W. McCuaig, Pólya's permanent problem, Electron. J. Combin. 11 (2004) R79.

[21] W. McCuaig, N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, in: Proc. Symp. on the Theory of Computing, STOC '97, 1997 (extended abstract).

[22] Y. Ohta, RIMS Kokyuroku, Kyoto Univ. 822 (1993) 197 (in Japanese).

[23] S. Okada, An elliptic generalization of Schur's Pfaffian identity, Adv. Math. 204 (2006) 530–538.

[24] S. Okada, Applications of minor-summation formulas to rectangular-shaped representations of classical groups, J. Algebra 205 (1998) 337–367.

[25] J. Propp, Enumeration of matchings: Problems and progress, in: L. Billera, A. Björner, C. Greene, R. Simeon, R.P. Stanley (Eds.), New Perspectives in Geometric Combinatorics, Cambridge Univ. Press, Cambridge, 1999, pp. 25–91.

[26] J. Propp, Generalized domino-shuffling, Theoret. Comput. Sci. 303 (2003) 267–301.

[27] N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, Ann. of Math. 150 (1999) 929–975.

[28] H. Srinivasan, Decomposition formulas for Pfaffians, J. Algebra 163 (1994) 312–334.

[29] J.R. Stembridge, Nonintersecting paths, Pfaffians, and plane partitions, Adv. Math. 83 (1990) 96–131.

[30] H.W.L. Tanner, A theorem relating to Pfaffians, Messenger Math. 8 (1878) 56–59.

[31] S. Tsujimoto, R. Hirota, Pfaffian representation of solutions to the discrete BKP hierarchy in bilinear form, J. Phys. Soc. Japan 65 (1996) 2797–2806.

[32] W.T. Tutte, The factorization of linear graphs, J. London Math. Soc. 22 (1947) 107–111.

[33] W. Wenzel, Geometric algebra of Δ-matroids and related combinatorial geometries, Habilitationsschrift, Bielefeld, 1991.

[34] W.G. Yan, Y.-N. Yeh, F.J. Zhang, Graphical condensation of plane graphs: A combinatorial approach, Theoret. Comput. Sci. 349 (2005) 452–461.

[35] W.G. Yan, F.J. Zhang, Enumeration of perfect matchings of graphs with reflective symmetry by Pfaffians, Adv. in Appl. Math. 32 (2004) 175–188.

[36] W.G. Yan, F.J. Zhang, Graphical condensation for enumerating perfect matchings, J. Combin. Theory Ser. A 110 (2005) 113–125.
[37] W.G. Yan, F.J. Zhang, Enumeration of perfect matchings of a type of Cartesian products of graphs, Discrete Appl. Math. 154 (2006) 145–157.